

Sinyaller ve Sistemler

Ders Notu

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Ders Konusu:

Sinyaller ve Sistemler, veri işleme, iletişim, görüntü işleme, savunma elektroniği, tüketici elektroniği dahil olmak üzere birçok farklı alanda mühendislik sistemlerinin ayrılmaz bir parçasını oluşturan, analog ve dijital sinyal işlemede temel niteliğindedir. Sürekli ve kesikli zaman sinyalleri ve sistemleri için temel kavramlar sunulacak ve birleştirilecektir. Sinyal ve sistemler hem zaman hem de frekans alanları için geliştirilmiştir. Bu temsiller, Fourier dönüşümü ve ayrıntılı olarak incelenen genellemeleri ile ilişkilidir. Hem analog hem de dijital sistemler için filtreleme ve filtre tasarımı, modülasyonu ve örneklemenin yanı sıra hem analog hem de dijital sistemler için geri bildirim sistemlerinin temel kavramlarının gösterilmesi ve gösterilmesi tartışılır ve gösterilir.

Ders Programları:

Matematiksel Kavramlar: Kompleks Sayılar, Homojen Diferansiyel Denklemler.

Sürekli Zamanlı İşaretler ve LTI Sistemleri

Sinyallerin Sınıflandırılması, Faydalı Sinyaller

Zaman Vektörleri ve Sinüzoidler

Sinyaller ve Gürültü, Kanal Kapasitesi

Örnekleme ve Aliasing

Sistemlerin sınıflandırılması, Frekans yanıtları

Fourier dönüşümü

Laplace dönüşümü

Z-Dönüşüm

Ayrık Zamanlı Sistem Modelleri, Sistem analiz

Filtreleme, Modülasyon

Matlab uygulamaları

İşbirliği yapılacak disiplinler:

- Olasılık, İstatistiksel veri analizi: Kesirim yaparak karar verme
- Uygulamalı matematik: Matlab, Türev, Integral, Limit, Laplace dönüşüm, Z-dönüşüm, Sayısal matematik, Nümerik analiz, trigonometri, Algoritma ve matematiksel modelleme
- Kaos: Dnamik ve dinamik olmayan, lineer olmayan; Risk ve sapma analizi, Manipülasyon, süreçlerde sapma analizi

Systems, signals, mathematical models will be explained in the details: General properties of signals, Continuous-time and discrete-time signals. Energy and power for continuous & discrete-time signals. Signal transformations. Specific signal types. Representing signals in Matlab and Simulink

Giriş

Var olduğumuz kainat, sinyaller denizidir. Yaşayan küçük bir organizma, bir hücre ya da en karmaşık ve en akıllı canlı organizma (insan), çevresinden sürekli sinyalleri alır ve işler. Herhangi bir canlı organizmanın hayatta kalması, sinyallerin uygun şekilde işlemesine ve tepki vermesine bağlıdır. Beynimiz çevresi ile algılama, hissetme ve koklama dahil pek çok sinyal ile mesaj gönderip almaktadır.

Sinyal bilgi taşıyan bir işarettir. Bu ders notunda, sinyallerin matematiksel temsilleri anlatılacak. Çevremizdeki sinyallere örnekler mi arıyorsunuz? İnsan sesi, kuşların cıvıltıları, duman sinyalleri, jestler (işaret dili), çiçek kokularıdır. Vücut işlevlerimizin çoğu kimyasal sinyallerle düzenlenir, kör insanlar dokunma hissini kullanır. Arılar dans modelleriyle birbirleriyle iletişim kurarlar. Ses uzayda yayılır mı? Uzay neden karanlık?

Teknolojik sinyallerinin bazı örnekleri, bir telefon teli içindeki akım ve gerilim, bir verici antenden çıkan elektromanyetik dalga, bir fiber kablonun kılındaki ışık yoğunluğunun değişmesidir. Böylece neredeyse sonsuz bir çeşitlilikte sinyaller olduğunu ve sinyalleri bir yerden başka bir yere taşıyan yol ve taşıyıcılar olduğunu görüyoruz.

Sinyal: Bir veya daha fazla değişkenlerin gerçek veya karmaşık fonksiyonudur. Fonksiyon tek bir değişkene bağlı olduğunda, sinyal bir boyutludur. Bir konuşma sinyali, günlük maksimum sıcaklık, bir yerdeki yıllık yağış, tek boyutlu bir sinyal örneğidir. Fonksiyon iki veya daha fazla değişkene bağlı olduğunda, sinyalin çok boyutlu olduğu söylenir. Bir görüntü, iki boyutlu sinyali, iki boyutu temsil eden dikey ve yatay koordinatları temsil etmektedir. Fiziksel dünyamız üç mekansal ve bir zamansal dört boyutludur.

Örneğin Morötesi ya da ultraviyole (kısaca UV) ışınım, dalga boyu 100 ile 400 nm arasındaki ışına denir. Gözümüz, 400 ile 700 nm dalga boyları arasına duyarlıdır ve bunun dışındaki ışınımı algılayamaz. Gecenin bir vakti kalkıp, çöle veya çöl iklimli bir yere gidin ve ultraviyole algılayıcıyı açın. Algılama ekranda canlı bir mavi-yeşil renkle parlayan akrepleri göreceksiniz. UV ışını altında vücutlarında algılayıcı olarak görev yapan kimyasalların bir kısmı bozunur ve sözü edilen renklerin etrafa saçılmasına neden olurlar.

Kızılötesi ışınım ise cisimlerin sıcaklığını uzaktan belirlemeye yarar. Termal görüntüleme genelde askeri ve sanayi amaçlarla kızılötesi kameralar olarak tüketici pazarına da girmiş bulunmaktadır. Geniş ve belirgin sıcaklıktaki soğuk ortamlarda (deniz, orman, çöl, karlı dağ, bozkır) havadan arama kurtarma çalışmalarında insan gözünden kolay kaçan sıcak insan vücudunun, ve konaklama ateş yerinin termografi ile daha kolay farkına varılır. Kızılötesi

ışınımın dalga boyu 750 nanometre ile 1 mikrometre arasındadır. Normal sıcaklığındaki insan vücudu 10 mikrometre civarında ışıma yapar.

1. Sinyaller

Sinyal; genellikle zaman içinde deterministik (belli bir kurala göre) olarak ya da rasgele (belli bir kurala ya da analitik denkleme göre tanımlanamayan) değerler dizisidir. Örnek vermek gerekirse bir sıcaklık sensörünün oda içinde belirli zaman aralıklarında ölçtüğü sıcaklık değerleri bir sinyaldir. Örnekte verilen sinyali analiz etmek istersek bu sıcaklık değerlerini oda içinde ölçülen en yüksek sıcaklık değeri ile normalize edersek (en yüksek sıcaklık değerine bölersek) sıfırın üzerindeki sıcaklık değerleri için 0 ile 1 arasında değişen bir sinyal elde etmiş oluruz. Bu sinyal için bağımsız değişken zamanı, sinyal ise bu zamanlarda oda içinde ölçülen en yüksek sıcaklık değerine normalize edilmiş oran değerlerini verir. Eğer sıcaklık ölçümlerini sürekli olarak alıyorsak sinyalimiz zamanın sürekli bir fonksiyonu, günde bir kere alıyorsa sinyalimiz zamanın kesikli bir fonksiyonu (her gün için bir değer ve diğer ölçüm zamanı gelene kadar herhangi bir değer yok) olarak tanımlanacaktır.

Bir sinyali $x(t)$ notasyonu ile ifade edilirse x bir sinyali, t bağımsız değişken olan zamanı, yumuşak parantezler ise zaman sinyalinin sürekli olduğunu gösterecektir. Peki, bağımsız zaman değişkeni t sürekli olmakla birlikte, sinyalin aldığı değer $x(t)$ süreklidir.

Bağımsız değişkeni t yerine n ile göstermek sürekli zaman sinyalini kesikli hale getirir. Sinyali kesikli zaman sinyali yapan notasyon $x[n]$, $n = \dots, -2, -1, 0, 1, 2, \dots; n \in \mathbb{Z}$ gösterimidir. Sinyal x ile gösterilmekte, n bağımsız değişkenin kesikli zaman değişkeni olduğunu göstermekte, köşeli parantezler ise sinyalin kesikli zaman sinyali olduğunu ifade etmektedir. Bağımsız zaman değişkeni n kesikli olmakla birlikte, sinyalin aldığı değer $x[n]$ değeri de kesiklidir.

Sinyal zamanda kesikli olmasına rağmen genlik değeri sürekli (irrasyonel dâhil) hatta karmaşık (complex) bir sayı değeri alabilir. Bu nedenle $x[n]$ sinyallerini zamanda kesikli ancak genlik değerinde sürekli olmak üzere kısaca kesikli zaman sinyalleri (discrete time signals) olarak adlandırmaktayız. Bu sinyaller literatürde sayısal (digital) sinyaller olarak adlandırılmaktadır. Kesikli zaman (discrete time) sinyallerden sayısal (digital) sinyallere geçiş (genlik ekseninin sürekli değerlerden kesikli değerlere çevrilmesi) nicemleme (quantization) işleminin konusudur. Bir $f(t)$ sinyali zamanın bir fonksiyonudur. Gerilim $v(t)$ veya akım $i(t)$ olabilir.

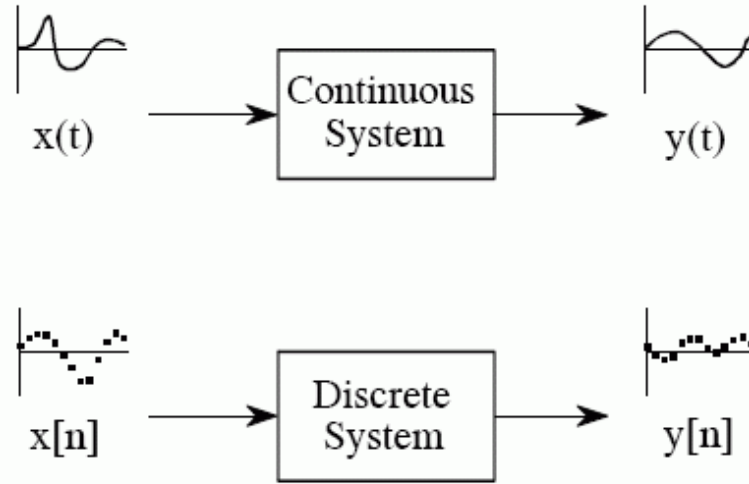


FIGURE 5-1

Terminology for signals and systems. A system is any process that generates an output signal in response to an input signal. Continuous signals are usually represented with parentheses, while discrete signals use brackets. All signals use lower case letters, reserving the upper case for the frequency domain (presented in later chapters). Unless there is a better name available, the input signal is called: $x(t)$ or $x[n]$, while the output is called: $y(t)$ or $y[n]$.

Sinyal örnekleri:

- Elektriksel sinyaller; devredeki voltajlar ve akımlar
- Akustik sinyaller, Zamanla akustik basınç (ses), Mekanik sinyaller
- Video sinyalleri; Zaman içinde bir pikselin (kamera, video) yoğunluğu
- Sıcaklık, hız gibi genliklerin zamanla değişimi
- Koku
- Kuantum parçacık yayını, Radyoaktif maddeler veya izotoplar
- Elektromanyetik yayını
- Kimyasal yayını – hormon
- Çekim Kuvvetleri
- Enerji kaynakları; ışık

Sistemler istenilen çıkış sinyallerini üretmek için giriş sinyallerini işler. Bir sistem bir sinyali girdi olarak alır ve başka bir sinyale dönüştürür.

Signals:

A signal is defined as a function of one or more variables which conveys information.

A signal is a physical quantity that varies with time in general, or any other independent variable.

It can be dependent on one or more independent variables.

Ex. $x_1(t) = 20t$ One-dimensional signal
 $x_2(t) = 20t^2$

$g(x,y) = x+3xy+2y^2$ Two-dimensional signal

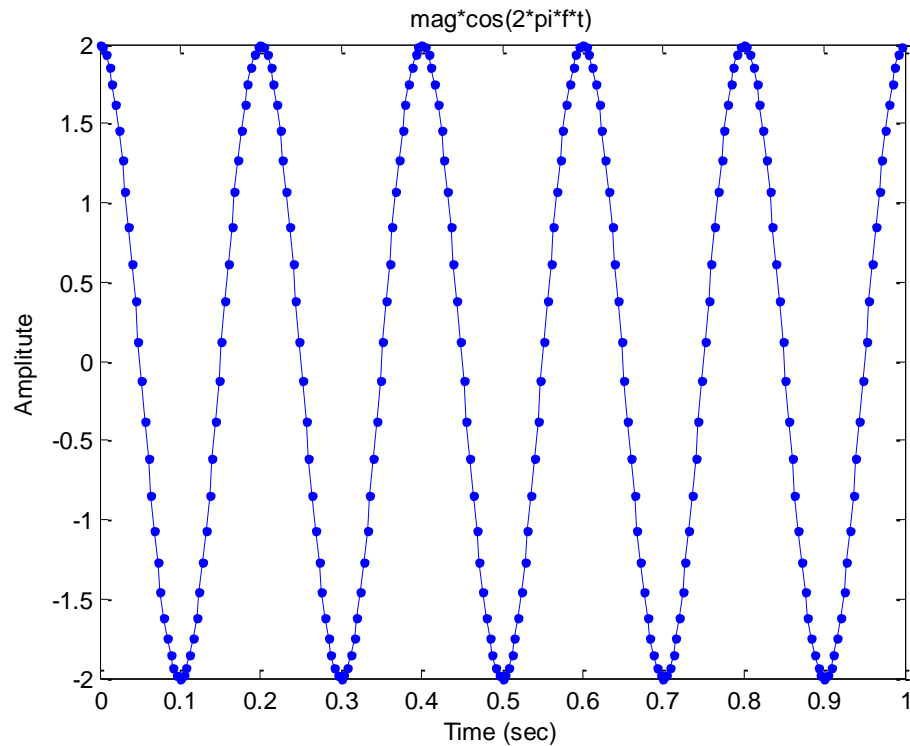
$x(t) = \sum_{i=1}^N A_i(t)\sin[2\pi f_i(t)t + \theta_i(t)];$ Multi-dimensional
 $A_i(t) =$ time varying i^{th} amplitude
 $f_i(t) =$ time varying i^{th} frequency
 $\theta_i(t) =$ time varying i^{th} phase

*Dimensions- Based on the number of independent variables.

Örnek:

```
clear all
close all
mag = 2;            % magnitude (arbitrary units)
f = 1000;          % frequency in Hz
Ps=50;            % number of sampling on a periot
samp = f*Ps;      % sampling rate in Hz
M=2;    %    Displaying number of period
del=1/samp;
t            = 0:del:M*Ps*del;    % time
N            = length(t)
x = mag*cos(2*pi*f*t);      % the signal equation
figure
plot(t,x, '-');
```

```
xlabel('Time (sec)');  
ylabel('Amplitude');  
title('mag*cos(2*pi*f*t)')
```



Waveform Generation: Time Vectors and Sinusoids

Consider generating data with a 1000 Hz sample frequency, for example. An appropriate time vector is

```
t = (0:0.001:1)';
```

where (:) creates a 1001-element row vector that represents time running from 0 to 1 seconds in steps of 1 ms. The transpose operator (') changes the row vector into a column; the semicolon (;) tells MATLAB to compute, but not display, the result.

Given t, you can create a sample signal y consisting of two sinusoids, one at 50 Hz and one at 120 Hz with twice the amplitude.

```
y = sin(2*pi*50*t) + 2*sin(2*pi*120*t);
```

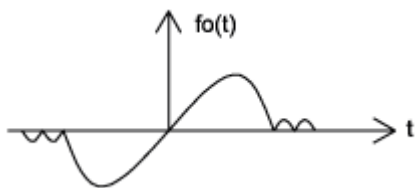
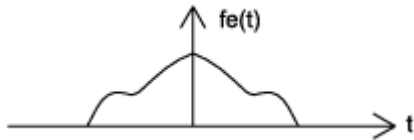
The new variable y , formed from vector t , is also 1001 elements long. You can add normally distributed white noise to the signal and plot the first 50 points:

```
yn = y + 0.5*randn(size(t));
```

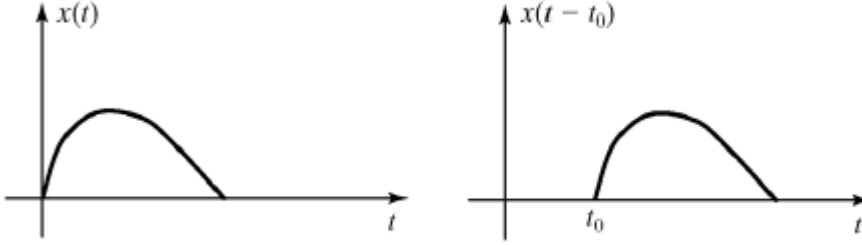
```
plot(t(1:50),yn(1:50))
```

Çift (Even) ve Tek (Odd) Sinyaller

Tek (Odd) ve çift (Even) sinyaller; Çift sinyaller, dikey eksene göre simetrik olan sinyallerdir ve matematiksel olarak $f(t)=f(-t)$ şeklinde ifade edilirler. Tek sinyaller ise $f(t)=-f(-t)$ biçimindedir.



Zaman Öteleme: Bir $x(t)$ işaretini verilen sabit bir t_0 zamanı kadar öteleme, veya geciktirme, $x(t-t_0)$ şeklinde bir işaret üretir.



Şekil İşaretin ötelenmesi

Çift sinyalin fonksiyonu, $x(t) = x(-t)$ olarak tanımlanır. Tek sinyalin fonksiyonu, $x(t) = -x(t)$ olarak tanımlanır. Bir çift sinyal ve tek sinyalin çarpımı ne olur? Bir çift sinyal ve tek sinyalin toplamı ne olur?

Explanation The product of an even and odd signal is an odd signal. But product or sum of two even signals is even signal.

1.1. Periyodik Sinyaller

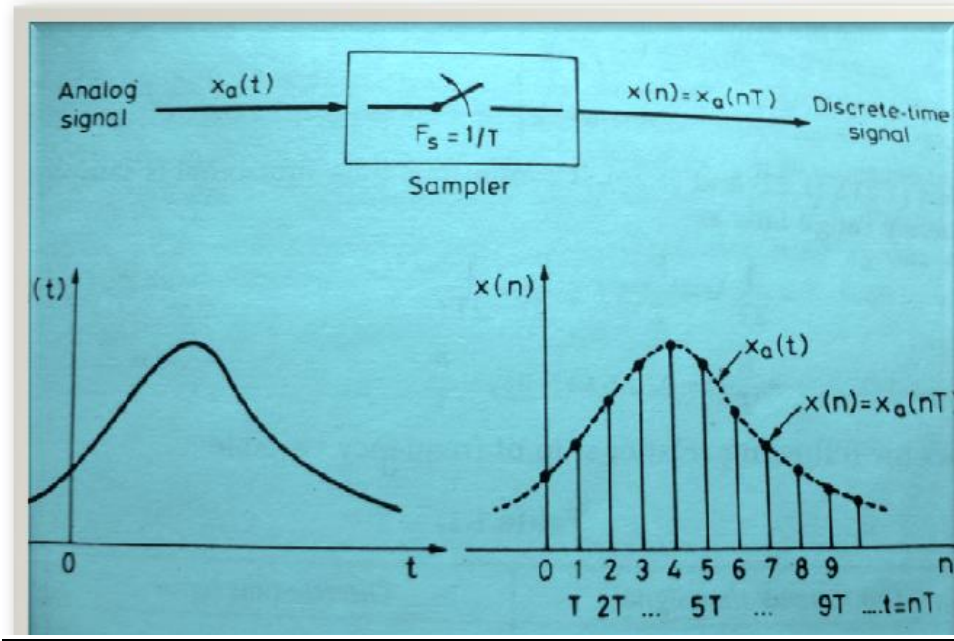
Some important classifications of signals

- Analog vs. Digital signals: as stated in the previous lecture, a signal with a magnitude that may take any real value in a specific range is called an analog signal while a signal with amplitude that takes only a finite number of values is called a digital signal.
- Continuous-time vs. discrete-time signals: continuous-time signals may be analog or digital signals such that their magnitudes are defined for all values of t , while discrete-time signal are analog or digital signals with magnitudes that are defined at specific instants of time only and are undefined for other time instants.
- Periodic vs. aperiodic signals: periodic signals are those that are constructed from a specific shape that repeats regularly after a specific amount of time T_0 , [i.e., a

periodic signal $f(t)$ with period T_0 satisfies $f(t) = f(t+nT_0)$ for all integer values of n , while aperiodic signals do not repeat regularly.

- Deterministic vs. probabilistic signals: deterministic signals are those that can be computed beforehand at any instant of time while a probabilistic signal is one that is random and cannot be determined beforehand.
- Energy vs. Power signals: as described below.

Periodic sampling of an analog signal



Nyquist teoremi: Bir analog sinyalin örnek alınıp orjinalitesini bozulmaması için $f_s \geq 2 \cdot B_w$ olmalıdır.

Periodic Signals

Zaman içinde kendini tekrar eden sinyallerdir.

$$X(t) = A \sin(\omega t + \phi)$$

A: Genlik; ω : açısal frekans; f: frekans (Hz=1/sec); T:periyod (sec)

$$\omega = 2\pi f, \quad f = 1/T, \quad T = 2\pi/\omega$$

- Bir işaretin periyodik olabilmesi için, $T = m \frac{2\pi}{\omega}$, m tamsayı olmalıdır.
- Birden fazla sinyalin toplamının periyodik olup olmadığını belirlemek için,

$$mT_1 = kT_2 = T$$

$$\frac{k}{m} = \frac{T_1}{T_2} = \text{kesirli sayı}$$

Kesirli yazılmayan sayılar irasyonel sayılardır ve periyodik olamazlar.

T₁/T₂ oranında, m=payda, k=pay olarak alınır.

X [n] = 2 cos (2n) is periodic or not?

Explanation: The given signal x [n] is non periodic as it doesn't satisfy the equation $\omega = 2\pi m/N$; where, N is fundamental period and m is an integer.

Fundamental frequency x[n] is given by $\omega = 2\pi/N$. Fundamental frequency is the smallest value of N which satisfies the equation. Where N is a positive integer.

Periyodik Fonksiyonlar :

$f:A \rightarrow B$ bir fonksiyon olsun. $\forall x \in A$ için $f(x+T) = f(x)$ eşitliğini sağlayan bir T gerçekte bir sayı varsa, f fonksiyonuna periyodik fonksiyon, T gerçekte bir sayısına da f' nin bir periyodu denir. T gerçekte bir sayısının en küçüğüne ise esas periyodu denir. Buradan hareketle;

$k \in Z$ olmak üzere $\forall \alpha \in \mathbb{R}$ için;

$\cos(\alpha + k.2\pi) = \cos \alpha$ ve $\sin(\alpha + k.2\pi) = \sin \alpha$ olduğundan sinüs ve kosinüs fonksiyonlarının periyodu $k.2\pi$ ve esas periyodu 2π dir.

Aynı şekilde;

$k \in Z$ olmak üzere $\alpha \neq \pi/2 + k\pi$ ve $\alpha \in \mathbb{R}$ için $\tan(\alpha + k.\pi) = \tan \alpha$

$k \in Z$ olmak üzere $\alpha \neq k\pi$ ve $\alpha \in \mathbb{R}$ için $\cot(\alpha + k.\pi) = \cot \alpha$ olduğundan tanjant ve kotanjant fonksiyonlarının periyodu $k.\pi$ ve esas periyodu π dir.

$$f(x) = \sin^m(ax+b) \quad \text{ve} \quad f(x) = \cos^m(ax+b)$$

$$\begin{array}{ll} m \text{ tek ise} & T = \frac{2\pi}{|a|} \\ m \text{ çift ise} & T = \frac{\pi}{|a|} \end{array}$$

An analog signal $x(t)$ is periodic if

- it is defined for all possible values of t , $-\infty < t < \infty$, and
- there is a positive real value T_0 , the *period* of $x(t)$, such that

$$x(t + kT_0) = x(t)$$

for any integer k .

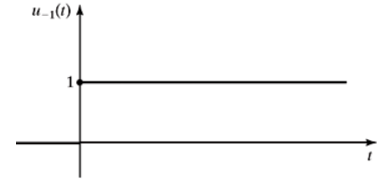
The period of $x(t)$ is the smallest possible value of $T_0 > 0$ that makes the periodicity possible. Thus, although NT_0 for an integer $N > 1$ is also a period of $x(t)$ it should not be considered the period.

$x(t) = A \cos(2\pi f_0 t + \theta)$ ve $x(t) = Ae^{j(2\pi f_0 t + \theta)}$ işaretleri gerçel ve karmaşık periyodik işaretlere örnek olarak verilebilir. Her iki işaretin periyodu $T_0 = \frac{1}{f_0}$

dir. Şekilde gösterilen

$$u_{-1}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad (2.1.5)$$

birim basamak fonksiyonu ise periyodik olmayan işaretlere bir örnek olarak verilebilir.



Şekil Birim basamak işareti.

Example

Let $x(t) = e^{j2t}$ and $y(t) = e^{j\pi t}$, and consider their sum $z(t) = x(t) + y(t)$, and their product $w(t) = x(t)y(t)$. Determine if $z(t)$ and $w(t)$ are periodic, and if so, find their periods. Is $p(t) = (1 + x(t))(1 + y(t))$ periodic?

Solution

According to Euler's identity,

$$x(t) = \cos(2t) + j \sin(2t)$$

$$y(t) = \cos(\pi t) + j \sin(\pi t)$$

indicating $x(t)$ is periodic of period $T_0 = \pi$ (the frequency of $x(t)$ is $\Omega_0 = 2 = 2\pi/T_0$) and $y(t)$ is periodic of period $T_1 = 2$ (the frequency of $y(t)$ is $\Omega_1 = \pi = 2\pi/T_1$).

For $z(t)$ to be periodic requires that T_1/T_0 be a rational number, which is not the case as $T_1/T_0 = 2/\pi$. So $z(t)$ is not periodic.

The product is $w(t) = x(t)y(t) = e^{j(2+\pi)t} = \cos(\Omega_2 t) + j \sin(\Omega_2 t)$ where $\Omega_2 = 2 + \pi = 2\pi/T_2$ so that $T_2 = 2\pi/(2 + \pi)$, so $w(t)$ is periodic of period T_2 .

The terms $1 + x(t)$ and $1 + y(t)$ are periodic of period $T_0 = \pi$ and $T_1 = 2$, and from the case of the product above, one would hope this product be periodic. But since $p(t) = 1 + x(t) + y(t) + x(t)y(t)$ and $x(t) + y(t)$ is not periodic, then $p(t)$ is not periodic.

Örnek:

$$x(t) = 2 \cos(2\pi t)$$

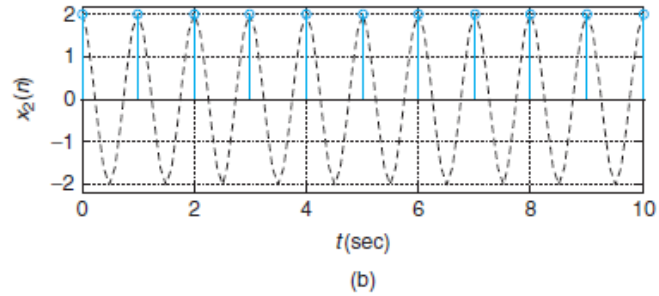
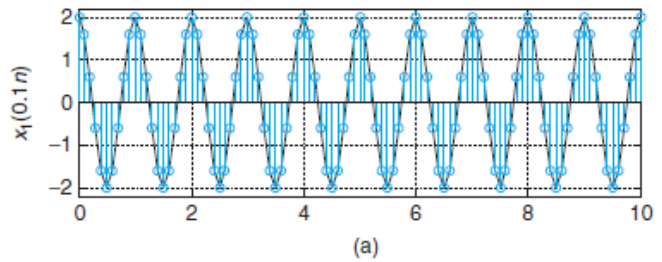
for $0 \leq t \leq 10$ sec. If we sample it every $T_{s1} = 0.1$ sec, the analog signal becomes the following sequence:

$$x_1[n] = x(t)|_{t=0.1n} = 2 \cos(2\pi n/10) \quad 0 \leq n \leq 100$$

providing a very good approximation to the original signal. If, on the other hand, we let $T_{s2} = 1$ sec, then the discrete-time signal becomes

$$x_2[n] = x(t)|_{t=n} = 2 \cos(2\pi n) = 2 \quad 0 \leq n \leq 10$$

Sampling an analog sinusoid $x(t) = 2 \cos(2\pi t)$, $0 \leq t \leq 10$, with two different sampling periods, (a) $T_{s1} = 0.1$ sec and (b) $T_{s2} = 1$ sec, giving $x_1(0.1n)$ and $x_2(n)$. The sinusoid is shown by dashed lines. Notice the similarity between the discrete-time signal and the analog signal when $T_{s1} = 0.1$ sec, while they are very different when $T_{s2} = 1$ sec, indicating loss of information.



Ayrık Sinüzoidaller

- Analog sinüzoidaller her zaman periyodiktir.
- Fakat ayrık sinüzoidallerin zarfları periyodik olmakla beraber işaretler her zaman değildir.

$$x(t) = A \cos(\Omega_0 t + \theta)$$

$$x(n) = A \cos(\omega_0 n + \phi)$$

$$T_0 = 2\pi / \Omega_0 \text{ sn} \rightarrow \text{periyot}$$

$$\Omega_0 \text{ radian / sn} \rightarrow \text{frekans}$$

$$A \rightarrow \text{genlik}$$

$$\theta \text{ radian} \rightarrow \text{faz}$$

işareti ancak

$$x(n + N) = A \cos(\omega_0 (n + N) + \phi) = x(n)$$

eşitliğini sağlayan N , pozitif tamsayı ise periyodiktir.

$$N = \frac{2\pi}{\omega_0}$$

Bu sinyalin bir periyodunda $P_s=20$ örnek alınmak istenmektedir. Örnekleme periyodu, $T_s=T/P_s$, ise, örnekleme frekansı, $f_s=1/T_s$ den f_s değerini bulunur.

Ayrık Sinüzoidaller Örnekler

1. N 'nin örnekleme periyodu ile ilişkisi

$$x(t) = A \cos(\Omega_0 t) \quad \xrightarrow{T_s} \quad x(T_0 n) = A \cos(\Omega_0 T_0 n) = A \cos(\omega_0 n)$$

$$\Omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

f_0 Hz \rightarrow işaretin frekansı

T_0 sn \rightarrow işaretin periyodu

T_s sn \rightarrow örnekleme periyodu

$$\omega_0 = \Omega_0 T_s = 2\pi \frac{T_s}{T_0}$$

Ayrık işaretin periyodik olması için

$$N = \frac{2\pi}{\omega_0} = \frac{T_0}{T_s}$$

ifadesi **tamsayı** olmalıdır.

Ayrık Sinüzoidaller Örnekler

2. $x(n) = 2 \cos\left(\frac{2\pi n}{8} + \frac{\pi}{4}\right)$ işareti periyodik midir ?

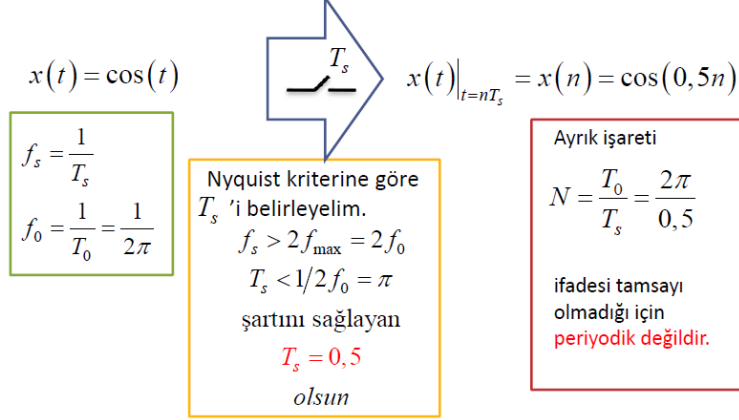
Evet , periyot $N=8$ pozitif tam sayıdır.

Sağlama:

$$x(n+8) = 2 \cos\left(\frac{2\pi(n+8)}{8} + \frac{\pi}{4}\right) = 2 \cos\left(\frac{2\pi n}{8} + 2\pi + \frac{\pi}{4}\right) = x(n)$$

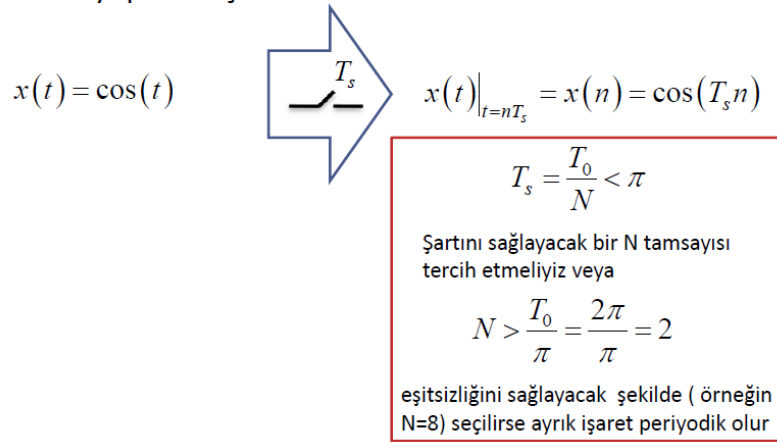
Ayrık Sinüzoidaller Örnekler

3. Frekansı 1 rad/sn olan analog sinüzoidten periyodik olmayan ayrık sinüzoid üretelim.



Ayrık Sinüzoidaller Örnekler

4. Önceki örnekteki ayrık işareti periyodik yapmak için:



Periyodik Ayırık Sinyaller

Eğer $x(n)$ işareti sonsuz uzunlukta ve k herhangi bir tamsayı ise

$$x(n) = x(n + kN)$$

işareti N periyoduyla periyodiktir.

Bir periyodik ayırık sinyalin Fourier serisi açılımı:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk}.$$

Fourier serisi katsayıları

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk}.$$

Periyodik Ayırık Sinyaller

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk}.$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk}.$$

- N periyoduyla periyodik bir ayırık sinyal, $[-\pi, \pi)$ veya eşdeğer $[0, 2\pi)$ aralığında N farklı harmoniğe sahiptir. $X(k)$ periyot N ile periyodiktir bu yüzden $X(k)$ sadece, m herhangi bir tam sayı olmak üzere $k=m, \dots, N-1+m$ aralığında hesaplanması yeterlidir.
- $X(k)$ Fourier seri katsayıları komplextir dolayısıyla genlik ve fazlarıyla tasvir edilmelidirler.
- Bir periyodik ayırık sinyalin Fourier serisi, o sinyalin bir periyodunun DTFT dönüşümü ve z -dönüşümü ile ilgilidir.

$$X(k) = X\left(e^{j\omega}\right)\Big|_{\omega=2\pi k/N} = X(z)\Big|_{z=e^{j2\pi k/N}}$$

Determining the periodicity of the sum of periodic signals

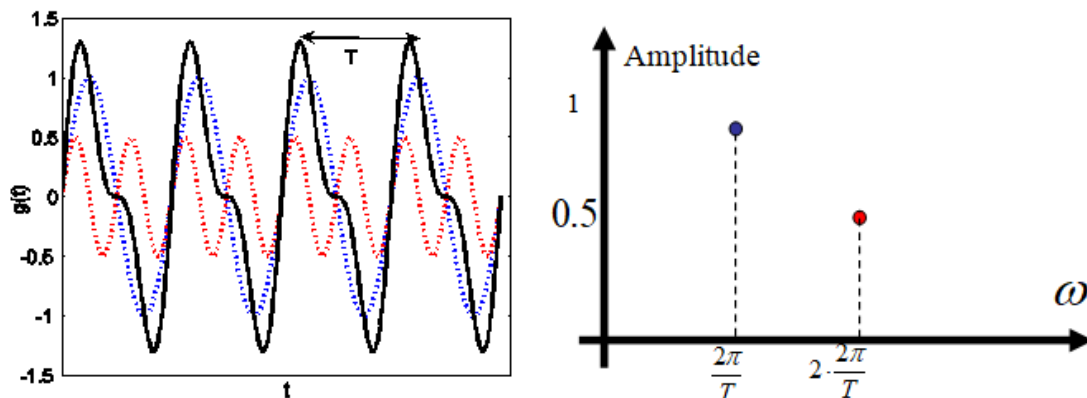
Let $x(t) = x_1(t) + x_2(t) + \dots + x_N(t)$ where $x_j(t)$ is a periodic signal with fundamental period T_j seconds, $j = 1, 2, \dots, N$.

To determine whether $x(t)$ is periodic, form the ratios T_1 / T_j , $j=2, \dots, N$. If every one of the ratios is a rational number, then $x(t)$ is periodic. Otherwise, it is not periodic.

If $x(t)$ is periodic, we determine its fundamental period as below:

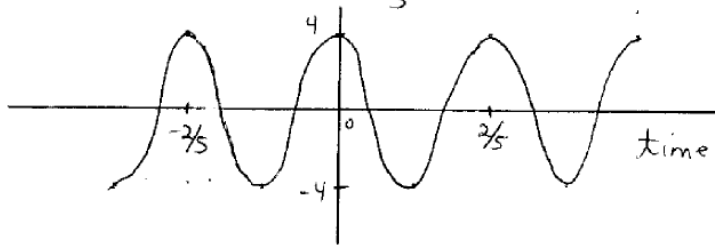
1. Convert the ratios T_1 / T_j , $j = 2, \dots, N$ to a ratio of integers, with common factors between numerator and denominator canceled out. Now the ratios will be of the form N_j / D_j , for $j=2, \dots, N$.
2. Find the least common multiple (LCM) of the D_j 's. Let this number be K .
3. The fundamental period of $x(t)$, T , is then given by

if $g(t) = \cos\left(\frac{2\pi}{T}t\right) + 0.5\cos\left(2 \cdot \frac{2\pi}{T}t\right)$, the time domain plot and the spectrum are shown as follows.



1. a) $x(t) = 4 \cos(5\pi t)$

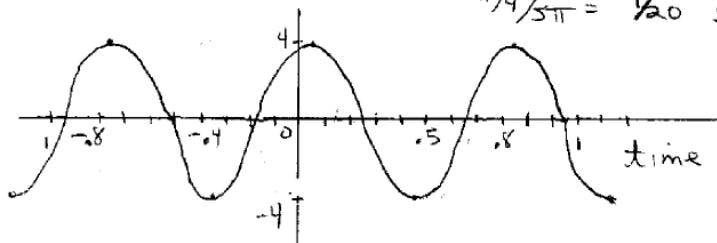
periodic with period $T = \frac{2\pi}{5\pi} = \frac{2}{5}$ sec
 to sketch: $A = 4$, $T = \frac{2}{5}$, $\theta = 0$



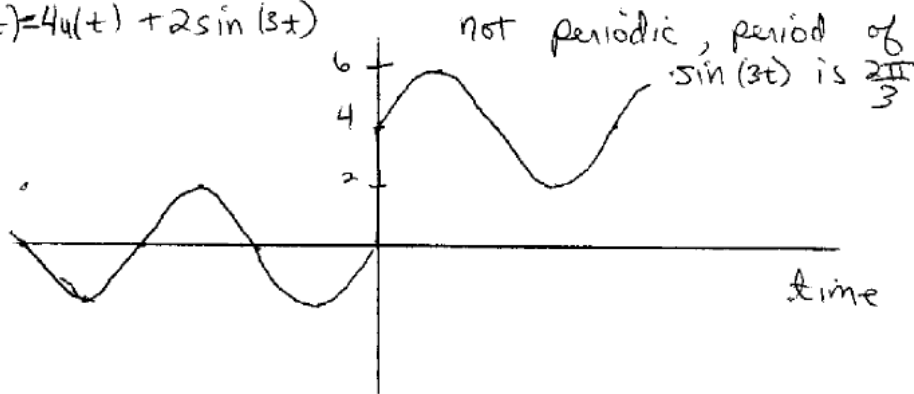
b) $x(t) = 4 \cos(5\pi t - \pi/4)$

periodic with period $T = \frac{2}{5}$ sec

to sketch: $A = 4$, $T = \frac{2}{5}$, $\theta = \pi/4$ so peak is at $\pi/4 / 5\pi = 1/20$ sec



c) $x(t) = 4u(t) + 2 \sin(3t)$



$$x(t) = \cos(3\pi t) + 2\cos(4\pi t)$$

$$T_1 = \frac{2\pi}{3\pi} = \frac{2}{3} \quad T_2 = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$$\frac{T_1}{T_2} = \frac{\frac{2}{3}}{\frac{1}{2}} = \frac{4}{3} \Rightarrow \text{periodic} \\ \text{with period } T = 2 \text{ sec.}$$

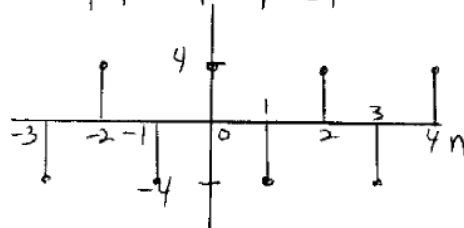
$$x[n] = 4\cos(\pi n)$$

periodic if $\Omega = \frac{2\pi g}{r}$ for some integers g or

In this case $\Omega = \pi$ so $g=1, r=2$ work.

So, this is periodic

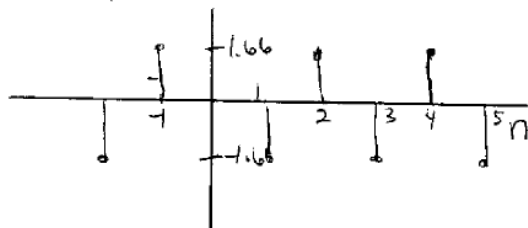
n	0	1	2	3
$4\cos(\pi n)$	4	-4	4	-4



Repeats every other value

$x[n] = 4\cos(\pi n - 2)$, periodic, repeats every other value of n

n	-1	0	1	2	3
$4\cos(\pi n - 2)$	1.66	-1.66	1.66	-1.66	1.66



$$a) x(t) = \cos(4t) + 2 \sin(8t)$$

$$T_1 = \frac{2\pi}{4} \quad T_2 = \frac{2\pi}{8}$$

$$\frac{T_1}{T_2} = \frac{\pi/2}{\pi/4} = \frac{2}{1} \Rightarrow \text{periodic with period } T_1 = \frac{\pi}{2} \text{ sec}$$

$$b) x(t) = 3 \cos(4t) + \sin(\pi t)$$

$$T_1 = \frac{2\pi}{4} \quad T_2 = \frac{2\pi}{\pi} = 2$$

$$\frac{T_1}{T_2} = \frac{\pi/2}{2} = \frac{\pi}{4} \neq \frac{q}{p} \quad \text{so not periodic}$$

Örnek:

The period of the signal $x(t) = 8 \sin\left(0.8\pi t + \frac{\pi}{4}\right)$ is

Period of $x(t)$,

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{0.8\pi} = 2.5 \text{ sec}$$

Soru-9: Aşağıdaki sinyallerin periyodik olup olmadığını belirleyin. Sinyal periyodik ise, temel periyodunu belirleyin.

(a) $x(t) = \sin\left(\frac{\pi}{5}t\right) \cos\left(\frac{2\pi}{7}t\right)$

(b) $x[n] = \left(\frac{1}{5}\right)^n \sin\left(\frac{\pi}{4}n\right)$

Periyodik olup olmadığını belirlemek için

$\sin(\omega_1 t) = \sin(2\pi f_1 t)$; $2\pi f_1 = \pi/5$ ise $f_1 = 1/10$ Hz olduğundan temel periyod $T_1 = 1/f_1 = 10$ bulunur

$\cos(\omega_2 t) = \cos(2\pi f_2 t)$; $2\pi f_2 = 2\pi/7$ ise $f_2 = 1/7$ Hz olduğundan temel periyod $T_2 = 1/f_2 = 7$ bulunur.

$X(t)$ 'nin temel periyodu $= T_1 * T_2 = 70$ bulunur.

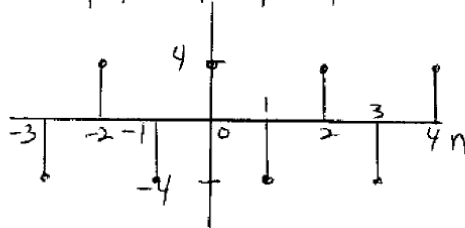
$x[n] = 4 \cos(\pi n)$

periodic if $\Omega = \frac{2\pi}{T} g$ for some integers g or

In this case $\Omega = \pi$ so $g=1, T=2$ work.

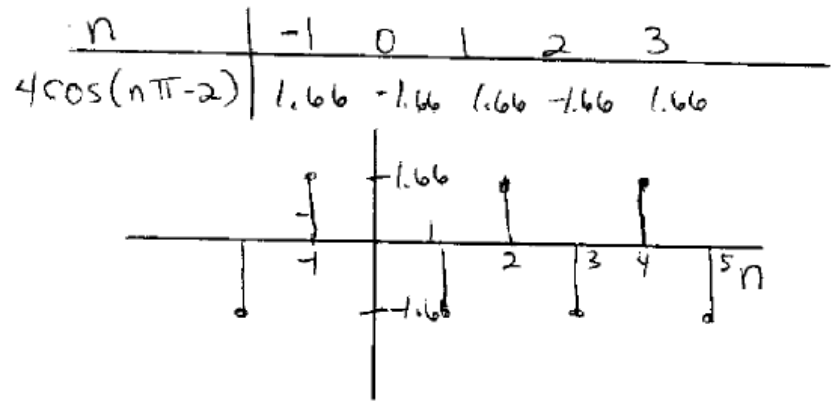
so, this is periodic

n	0	1	2	3
$4 \cos(\pi n)$	4	-4	4	-4



Repeats every other value

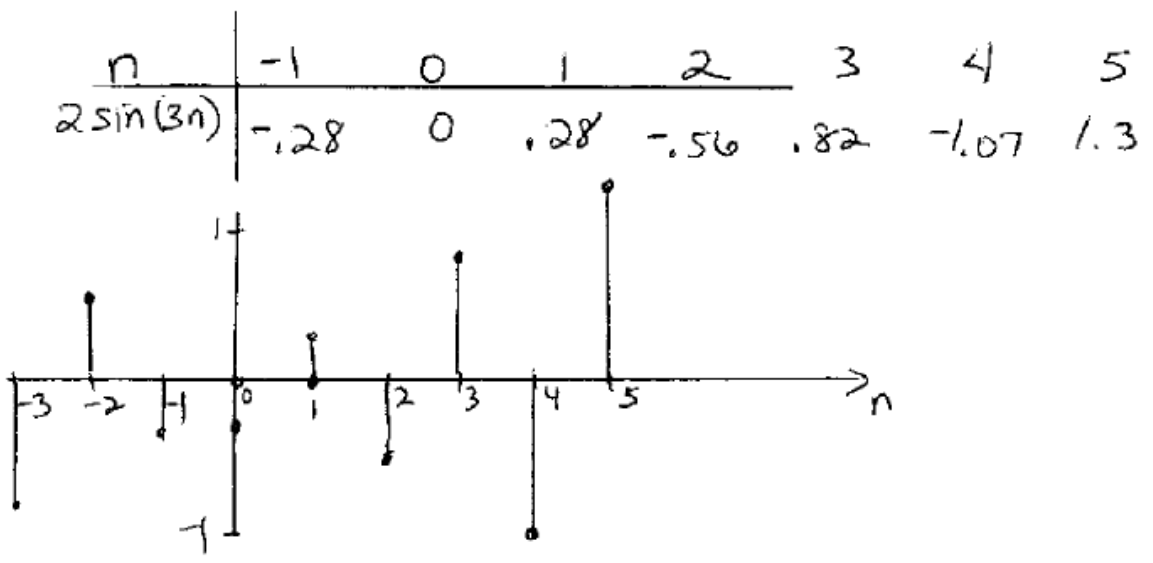
$x[n] = 4\cos(\pi n - 2)$, periodic, repeats every other value of n



$x[n] = 2\sin(3n)$

$\Omega = 3 \neq \frac{2\pi r}{T}$ for g, r integers

so, not periodic



$$x(t) = \cos(4t) + 2\sin(8t)$$

$$T_1 = \frac{2\pi}{4} \quad T_2 = \frac{2\pi}{8}$$

$$\frac{T_1}{T_2} = \frac{\pi/2}{\pi/4} = \frac{2}{1} \Rightarrow \text{periodic with period } T_1 = \frac{\pi}{2} \text{ sec}$$

$$x(t) = 3\cos(4t) + \sin(\pi t)$$

$$T_1 = \frac{2\pi}{4} \quad T_2 = \frac{2\pi}{\pi} = 2$$

$$\frac{T_1}{T_2} = \frac{\pi/2}{2} = \frac{\pi}{4} \neq \frac{p}{q} \text{ so not periodic}$$

$$x(t) = \cos(3\pi t) + 2\cos(4\pi t)$$

$$T_1 = \frac{2\pi}{3\pi} = \frac{2}{3} \quad T_2 = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$$\frac{T_1}{T_2} = \frac{2/3}{1/2} = \frac{4}{3} \Rightarrow \text{periodic with period } T = 2 \text{ sec.}$$

Soru:

$$(a) \quad x(t) = \cos\left(t + \frac{\pi}{4}\right)$$

$$(a) \quad x(t) = \cos\left(t + \frac{\pi}{4}\right) = \cos\left(\omega_0 t + \frac{\pi}{4}\right) \rightarrow \omega_0 = 1$$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 2\pi$.

$$(b) \quad x(t) = \sin \frac{2\pi}{3} t$$

$$(b) \quad x(t) = \sin \frac{2\pi}{3} t \rightarrow \omega_0 = \frac{2\pi}{3}$$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 3$.

$$(c) \quad x(t) = \cos \frac{\pi}{3} t + \sin \frac{\pi}{4} t = x_1(t) + x_2(t)$$

where $x_1(t) = \cos(\pi/3)t = \cos \omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 6$ and $x_2(t) = \sin(\pi/4)t = \sin \omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = 8$. Since $T_1/T_2 = \frac{6}{8} = \frac{3}{4}$ is a rational number, $x(t)$ is periodic with fundamental period $T_0 = 4T_1 = 3T_2 = 24$.

$$(d) \quad x(t) = \cos t + \sin \sqrt{2} t = x_1(t) + x_2(t)$$

where $x_1(t) = \cos t = \cos \omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 2\pi$ and $x_2(t) = \sin \sqrt{2} t = \sin \omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = \sqrt{2}\pi$. Since $T_1/T_2 = \sqrt{2}$ is an irrational number, $x(t)$ is nonperiodic.

1.2. Energy and Power Signals

There are in principle many different energy fields available from which to harvest energy:

- Radiation (light, solar, cosmic rays, electromagnetic radiation)
- Thermal
- Mechanical (potential, kinetic, elastic, fluid)
- Gravitational
- Chemical (battery, fuel cell, fossil fuels)
- Nuclear
- Magnetic (Magnetisation, currents etc)
- Electric

Güç ve Enerji nedir

Güç, iş yapma potansiyelidir. Güç, enerji üretme oranıdır. Bir başka ifadeyle, birim zamanda yapılan iştir. Birimleri kgm/s, W veya HP'dir. Güç, Watt (W) cinsinden ölçülür.

- 75 kgm/s = 1 HP'dir,
- 100 kgm/s = 1 KW'tır.

Enerji, "iş yapabilme yeteneğidir". İş yapıldığında, enerji bir formdan diğerine dönüşür. Enerji birimi joule (J)'dür. Enerji, farklı formlarda ve amaçlarla kullanılır. Enerji, güçtür; enerji, paradır, enerji iştir. Birimi kgm'dir.

Elektrikte, Watt (W) güç ölçüm birimidir. Amper x Volt = Watt.

Elektrikte enerji ölçüm birimi Watt-hour (Wh) dir. Saatlik üretilen ya da tüketilen elektrik enerjisi miktarıdır. 1 Kilowatt-hour (kWh) = 1000 Wh

- 1 kWh = 860 Kcal'dir veya 3415 Btu'dur.

1 Kilowatt-Hour = 1000 watts x 3600 seconds = 3.6×10^6 watt-second = 3.6×10^6 Joules

How much heat is produced by a human body?

Explanation: A man doing no or very little physical work needs about 2,000 kcal (or less) of energy in his daily food. The body converts this energy almost entirely into heat.

1 day = 24 x 60 x 60 s = 86,400 s 1 cal = 4.2 J

Hence, 2000 kcal/day = 2000 x 4.2 kJ/day = (8.4 MJ)/86600s = 100 J/s = 100 W

We see that a human body doing no work is equivalent to a heat source of about 100 W – the equivalent of a good bulb.

Güç: Anlık ve Normalize

Bir devrede Anlık Güç: $p(t) = v(t)i(t)$

Ohm kanunundan, $p(t) = \frac{v^2(t)}{R} = i^2(t)R$

Anlık Normalize Güç $R=1$ ohm alınarak bulunur. $p(t) = v^2(t) = i^2(t)$

$f(t)$ bir gerilim veya bir akım fonksiyonu olur ise $p(t) = f^2(t)$ olarak yazılır.

The total energy contained in and average power provided by a signal $f(t)$ (which is a function of time) are defined as

$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

and

$$P_f = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt$$

respectively.

For periodic signals, the power P can be computed using a simpler form based on the periodicity of the signal as

$$P_{\text{Periodic } f} = \frac{1}{T} \int_{t_0}^{T+t_0} |f(t)|^2 dt$$

where T here is the period of the signal and t_0 is an arbitrary time instant that is chosen to simplify the computation of the integration (to reduce the functions you have to integrate over one period).

Classification of Signals into Power and Energy Signals

Most signals can be classified into Energy signals or Power signals. A signal is classified into an energy or a power signal according to the following criteria

- a) Energy Signals: an energy signal is a signal with finite energy and zero average power ($0 \leq E < \infty, P = 0$),
- b) Power Signals: a power signal is a signal with infinite energy but finite average power ($0 < P < \infty, E \rightarrow \infty$).

Comments:

1. The square root of the average power \sqrt{P} of a power signal is what is usually defined as the RMS value of that signal.
2. Your book says that if a signal approaches zero as t approaches ∞ then the signal is an energy signal. This is in most cases true but not always as you can verify in part (d) in the following example.
3. All periodic signals are power signals (but not all non-periodic signals are energy signals).
4. Any signal f that has limited amplitude ($|f| < \infty$) and is time limited ($f = 0$ for $|t| > t_0$ for some $t_0 > 0$) is an energy signal as in part (g) in the following example.

Exercise 2–1: determine if the following signals are Energy signals, Power signals, or neither, and evaluate E and P for each signal

a) $a(t) = 3\sin(2\pi t), -\infty < t < \infty,$

This is a periodic signal, so it must be a power signal. Let us prove it.

$$\begin{aligned} E_a &= \int_{-\infty}^{\infty} |a(t)|^2 dt = \int_{-\infty}^{\infty} |3\sin(2\pi t)|^2 dt \\ &= 9 \int_{-\infty}^{\infty} \frac{1}{2} [1 - \cos(4\pi t)] dt \\ &= 9 \int_{-\infty}^{\infty} \frac{1}{2} dt - 9 \int_{-\infty}^{\infty} \cos(4\pi t) dt \\ &= \infty \quad \text{J} \end{aligned}$$

Notice that the evaluation of the last line in the above equation is infinite because of the first term. The second term has a value between -2 to 2 so it has no effect in the overall value of the energy.

Since $a(t)$ is periodic with period $T = 2\pi/2\pi = 1$ second, we get

$$\begin{aligned}
 P_a &= \frac{1}{1} \int_0^1 |a(t)|^2 dt = \int_0^1 |3\sin(2\pi t)|^2 dt \\
 &= 9 \int_0^1 \frac{1}{2} [1 - \cos(4\pi t)] dt \\
 &= 9 \int_0^1 \frac{1}{2} dt - 9 \int_0^1 \cos(4\pi t) dt \\
 &= \frac{9}{2} - \left[\frac{9}{4\pi} \sin(4\pi t) \right]_0^1 \\
 &= \frac{9}{2} \text{ W}
 \end{aligned}$$

So, the energy of that signal is infinite and its average power is finite (9/2). This means that it is a power signal as expected. Notice that the average power of this signal is as expected (square of the amplitude divided by 2)

b) $b(t) = 5e^{-2|t|}, -\infty < t < \infty,$

Let us first find the total energy of the signal.

$$\begin{aligned}
 E_b &= \int_{-\infty}^{\infty} |b(t)|^2 dt = \int_{-\infty}^{\infty} |5e^{-2|t|}|^2 dt \\
 &= 25 \int_{-\infty}^0 e^{4t} dt + 25 \int_0^{\infty} e^{-4t} dt \\
 &= \frac{25}{4} [e^{4t}]_{-\infty}^0 + \frac{25}{4} [e^{-4t}]_0^{\infty} \\
 &= \frac{25}{4} + \frac{25}{4} = \frac{50}{4} \text{ J}
 \end{aligned}$$

The average power of the signal is

$$\begin{aligned}
 P_b &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |b(t)|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |5e^{-2|t|}|^2 dt \\
 &= 25 \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^0 e^{4t} dt + 25 \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{T/2} e^{-4t} dt \\
 &= \frac{25}{4} \lim_{T \rightarrow \infty} \frac{1}{T} [e^{4t}]_{-T/2}^0 + \frac{25}{4} \lim_{T \rightarrow \infty} \frac{1}{T} [e^{-4t}]_0^{T/2} \\
 &= \frac{25}{4} \lim_{T \rightarrow \infty} \frac{1}{T} [1 - e^{-2T}] + \frac{25}{4} \lim_{T \rightarrow \infty} \frac{1}{T} [e^{-2T} - 1] \\
 &= 0 + 0 = 0
 \end{aligned}$$

So, the signal $b(t)$ is definitely an energy signal.

So, the energy of that signal is infinite and its average power is finite (9/2). This means that it is a power signal as expected. Notice that the average power of this signal is as expected (the square of the amplitude divided by 2)

$$\text{c) } c(t) = \begin{cases} 4e^{-3t}, & |t| \leq 5 \\ 0, & |t| > 5 \end{cases},$$

$$\text{d) } d(t) = \begin{cases} \frac{1}{\sqrt{t}}, & t > 1 \\ 0, & t \leq 1 \end{cases},$$

Let us first find the total energy of the signal.

$$\begin{aligned}
 E_d &= \int_{-\infty}^{\infty} |d(t)|^2 dt = \int_1^{\infty} \frac{1}{t} dt \\
 &= \ln[t]_1^{\infty} \\
 &= \infty - 0 = \infty \text{ J}
 \end{aligned}$$

So, this signal is NOT an energy signal. However, it is also NOT a power signal since its average power as shown below is zero.

The average power of the signal is

$$\begin{aligned}
 P_d &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |d(t)|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_1^{T/2} \frac{1}{t} dt \\
 &= \lim_{T \rightarrow \infty} \left(\frac{1}{T} \ln[t]_1^{T/2} \right) = \lim_{T \rightarrow \infty} \left(\frac{1}{T} \ln \left[\frac{T}{2} \right] - \frac{1}{T} \ln[1] \right) \\
 &= \lim_{T \rightarrow \infty} \left(\frac{1}{T} \ln \left[\frac{T}{2} \right] \right) = \lim_{T \rightarrow \infty} \left(\frac{\ln \left[\frac{T}{2} \right]}{T} \right)
 \end{aligned}$$

Using Le'hopital's rule, we see that the power of the signal is zero. That is

$$P_d = \lim_{T \rightarrow \infty} \left(\frac{\ln \left[\frac{T}{2} \right]}{T} \right) = \lim_{T \rightarrow \infty} \left(\frac{\frac{2}{T}}{1} \right) = 0$$

So, not all signals that approach zero as time approaches positive and negative infinite is an energy signal. They may not be power signals either.

e) $e(t) = -7t^2, \quad -\infty < t < \infty,$

f) $f(t) = 2 \cos^2(2\pi t), \quad -\infty < t < \infty.$

g) $g(t) = \begin{cases} 12 \cos^2(2\pi t), & -8 < t < 31 \\ 0, & \text{elsewhere} \end{cases}.$

1.7 Enerji ve Güç Sinyalleri

Sonsuz değer alan sinyallerin enerjileri de sonsuz olacağından bu sinyallerin gücünden bahsedilir ve bu sinyallere güç sinyali denir. Eğer sonsuz değer almıyor ve enerjisi bir reel sayıya eşit oluyorsa bu tür sinyallere de enerji sinyali denir.

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$
$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-\infty}^{\infty} |x(n)|^2$$

- $0 < E < \infty$, Enerji sinyali, böylece $P = 0$.
- $0 < P < \infty$, Güç sinyali, böylece $E = \infty$.
- Bu iki koşula uymayan sinyaller ne enerji sinyalidir ne de güç.

The signal energy in the signal $x(t)$ is

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt . \quad (1)$$

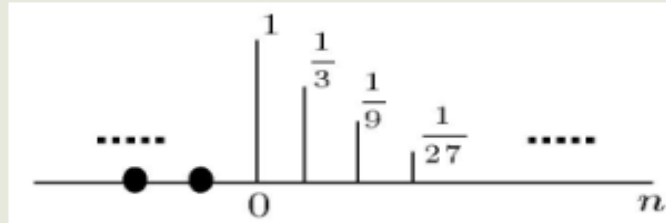
The signal power in the signal $x(t)$ is

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt . \quad (2)$$

If $0 < E < \infty$, then the signal $x(t)$ is called an energy signal. However, there are signals where this condition is not satisfied. For such signals we consider the power. If $0 < P < \infty$, then the signal is called a power signal. Note that the power for an energy signal is zero ($P = 0$) and that the energy for a power signal is infinite ($E = \infty$). Some signals are neither energy nor power signals.

Example : Compute the energy of the signal $x[n]$ given by

$$x[n] = \begin{cases} \left(\frac{1}{3}\right)^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$\begin{aligned} E_x &= \sum_{-\infty}^{\infty} x^2[n] = \sum_0^{\infty} x^2[n] = \sum_0^{\infty} \left(\frac{1}{3}\right)^{2n} \\ &= \sum_0^{\infty} \left(\frac{1}{9}\right)^n = \sum_0^{\infty} \left(\frac{1}{9}\right)^n = \frac{1}{1 - \frac{1}{9}} \\ &= \frac{9}{8} \end{aligned}$$

Örnek

$$x(t) = \begin{cases} 3 & |x| < 3 \\ 0 & \dots \end{cases}$$

şeklinde tanımlanmış olan işaretin enerjisini bulun

Çözüm

$$\mathcal{E}_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-3}^3 9 dt = 54.$$

olarak bulunur. Dolayısı ile bu işaret enerji işaretidir.

Örnek

$A \cos(2\pi f_0 t + \theta)$ işaretinin enerjisi;

$$\mathcal{E}_x = \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} A^2 \cos^2(2\pi f_0 t + \theta) dt = \infty.$$

olarak bulunur. Dolayısı ile bu işaret bir enerji işareti değildir. Ancak, işaretin gücü

$$\begin{aligned} P_x &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2 \cos^2(2\pi f_0 t + \theta) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{A^2}{2} [1 + \cos(4\pi f_0 t + 2\theta)] dt \\ &= \lim_{T \rightarrow \infty} \left[\frac{A^2 T}{2T} + \left[\frac{A^2}{8\pi f_0 T} \sin(4\pi f_0 t + 2\theta) \right]_{-T/2}^{T/2} \right] \\ &= \frac{A^2}{2} < \infty. \end{aligned}$$

olduğundan $x(t)$ bir güç işaretidir ve gücü $\frac{A^2}{2}$ dir.

Örnek

T_0 periyoduna sahip herhangi bir periyodik işaretin enerjisi;

$$\begin{aligned} \mathcal{E}_x &= \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} |x(t)|^2 dt \\ &= \lim_{n \rightarrow \infty} \int_{-\frac{nT_0}{2}}^{+\frac{nT_0}{2}} |x(t)|^2 dt \\ &= \lim_{n \rightarrow \infty} n \int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} |x(t)|^2 dt \\ &= \infty. \end{aligned}$$

Dolayısı ile *periyodik işaretler enerji işareti değildir.*

$$\begin{aligned} P_x &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \\ &= \lim_{n \rightarrow \infty} \frac{1}{nT_0} \int_{-\frac{nT_0}{2}}^{+\frac{nT_0}{2}} |x(t)|^2 dt \\ &= \lim_{n \rightarrow \infty} \frac{n}{nT_0} \int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} |x(t)|^2 dt \\ &= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} |x(t)|^2 dt. \end{aligned}$$



Periyodik işaretin güç içeriği yandaki gibi verilir. Bu sonucun anlamı herhangi bir periyodik işaretin güç içeriğinin, bir periyot içindeki ortalama güce eşit olduğudur.

Example

Compute the signal energy and signal power for the discrete-time signal

$$x(n) = \left(\frac{1}{4}\right)^n u(n).$$

We apply relationship (12.5)

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=0}^{\infty} \left| \left(\frac{1}{4}\right)^n \right|^2 = \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n.$$

The expression on the right hand side is a geometric series; hence, we have

$$E = \frac{1}{1 - \frac{1}{16}} = \frac{16}{15}.$$

Since $0 < E < \infty$, signal $x(n)$ is an energy signal, consequently, $P = 0$.

Example :

Let us consider a complex signal

$$x(t) = Ae^{j\omega_0 t}.$$

Signal $x(t)$ is periodic with period $T_0 = \frac{2\pi}{\omega_0}$; hence, it cannot be an energy signal.

To compute the signal power we use (12.3)

$$P = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |Ae^{j\omega_0 t}|^2 dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A^2 dt = A^2.$$

Example

Let us consider a signal

$$x(t) = e^{-t}u(t).$$

The energy of this signal is

$$E = \int_{-\infty}^{\infty} |e^{-t}u(t)|^2 dt = \int_0^{\infty} e^{-2t} dt = -\frac{1}{2}e^{-2t} \Big|_0^{\infty} = \frac{1}{2}.$$

The signal $x(t)$ is an energy signal. Since E is finite the signal power $P = 0$.

Determine the total energy and average power of the following signals.

$$(a) \ x(t) = \begin{cases} 2, & -3 \leq t \leq 3 \\ 5 - t, & 3 \leq t \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

$$(b) \ x(t) = \cos(t) \sin(t)$$

$$(c) \ x[n] = a^n u[-n], \quad |a| > 1$$

$$\begin{aligned} E &= \int_{t=-\infty}^{\infty} |x(t)|^2 dt \\ &= \int_{t=-3}^3 |x(t)|^2 dt + \int_{t=3}^5 |x(t)|^2 dt \\ &= \int_{t=-3}^3 (2)^2 dt + \int_{t=3}^5 (5-t)^2 dt \\ &= \int_{t=-3}^3 4 dt + \int_{t=3}^5 (25 + t^2 - 10t) dt \\ &= (4t) \Big|_{t=-3}^3 + \left(25t + \frac{1}{3}t^3 - 5t^2 \right) \Big|_{t=3}^5 \\ &= (4(3) - 4(-3)) + (25(5-3) + \frac{1}{3}(5^3 - 3^3) - 5(5^2 - 3^2)) \\ &= 24 + (50 + \frac{98}{3} - 80) \\ &= 24 + \frac{8}{3} \\ &= \frac{80}{3} \end{aligned}$$

Soru:

$$a(t) = 3\sin(2\pi t), \quad -\infty < t < \infty$$

$$\begin{aligned} E_a &= \int_{-\infty}^{\infty} |a(t)|^2 dt = \int_{-\infty}^{\infty} |3\sin(2\pi t)|^2 dt \\ &= 9 \int_{-\infty}^{\infty} \frac{1}{2} [1 - \cos(4\pi t)] dt \\ &= 9 \int_{-\infty}^{\infty} \frac{1}{2} dt - 9 \int_{-\infty}^{\infty} \cos(4\pi t) dt \\ &= \infty \quad \text{J} \end{aligned}$$

$$\begin{aligned} P_a &= \frac{1}{1} \int_0^1 |a(t)|^2 dt = \int_0^1 |3\sin(2\pi t)|^2 dt \\ &= 9 \int_0^1 \frac{1}{2} [1 - \cos(4\pi t)] dt \\ &= 9 \int_0^1 \frac{1}{2} dt - 9 \int_0^1 \cos(4\pi t) dt \\ &= \frac{9}{2} - \left[\frac{9}{4\pi} \sin(4\pi t) \right]_0^1 \\ &= \frac{9}{2} \quad \text{W} \end{aligned}$$

$$b(t) = 5e^{-2|t|}, \quad -\infty < t < \infty$$

$$\begin{aligned} E_b &= \int_{-\infty}^{\infty} |b(t)|^2 dt = \int_{-\infty}^{\infty} |5e^{-2|t|}|^2 dt \\ &= 25 \int_{-\infty}^0 e^{4t} dt + 25 \int_0^{\infty} e^{-4t} dt \\ &= \frac{25}{4} [e^{4t}]_{-\infty}^0 + \frac{25}{4} [e^{-4t}]_0^{\infty} \\ &= \frac{25}{4} + \frac{25}{4} = \frac{50}{4} \quad \text{J} \end{aligned}$$

$$\begin{aligned} P_b &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |b(t)|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |5e^{-2|t|}|^2 dt \\ &= 25 \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^0 e^{4t} dt + 25 \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{T/2} e^{-4t} dt \\ &= \frac{25}{4} \lim_{T \rightarrow \infty} \frac{1}{T} [e^{4t}]_{-T/2}^0 + \frac{25}{4} \lim_{T \rightarrow \infty} \frac{1}{T} [e^{-4t}]_0^{T/2} \\ &= \frac{25}{4} \lim_{T \rightarrow \infty} \frac{1}{T} [1 - e^{-2T}] + \frac{25}{4} \lim_{T \rightarrow \infty} \frac{1}{T} [e^{-2T} - 1] \\ &= 0 + 0 = 0 \end{aligned}$$

Soru:

$$x(t) = \begin{cases} t+1, & -1 \leq t < 0 \\ 1 & 0 \leq t < 1 \\ 0 & \text{elsewhere} \end{cases} .$$

a) Sinyalin enerjisini bulunuz.

$$\begin{aligned} & \int_{-1}^0 (t+1)^2 dt + \int_{-1}^0 (1)^2 dt \\ &= \int_0^1 (u)^2 du + 1 \\ &= \frac{1}{3} u^3 \Big|_0^1 + 1 \\ &= \frac{4}{3} \end{aligned}$$

Soru:

$x(t)$ sinyalinin enerjisi,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt .$$

formülü ile hesaplanır.

$x(t)$ sinyalinin gücü,

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt .$$

formülü ile hesaplanır.

$$x(t) = e^{-t}$$

Sinyalin enerjisi hesaplanırken,

$$E = \int_{-\infty}^{\infty} |e^{-t}u(t)|^2 dt = \int_0^{\infty} e^{-2t} dt$$

bağıntısı kullanılmaktadır. O halde sinyalin enerjisini hesaplayınız.

The energy of this signal is

$$E = \int_{-\infty}^{\infty} |e^{-t}u(t)|^2 dt = \int_0^{\infty} e^{-2t} dt = -\frac{1}{2} e^{-2t} \Big|_0^{\infty} = \frac{1}{2} .$$

The signal $x(t)$ is an energy signal. Since E is finite the signal power $P = 0$.

Soru: Aşağıdaki sinyallerin toplam enerjisini ve ortalama gücünü hesaplayın.

(a) $x[n] = 10 (u[n - 1] - u[n - 10])$

(b) $x(t) = 2e^{j\frac{\pi}{2}t}$

$$\begin{aligned} \text{(a) Energy} &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= \int_1^{10} 10^2 dt + 0 \\ &= 900 \end{aligned}$$

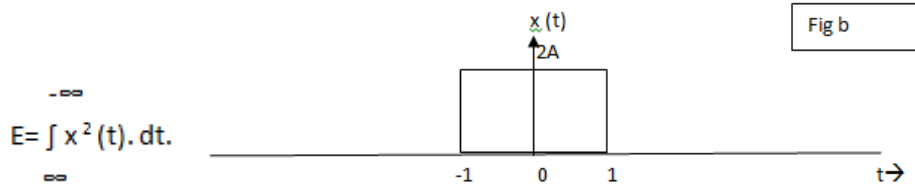
Since energy is finite, power is zero.

(b) The signal is a complex exponential with fundamental period $T_0 = 4$. Moreover $|x(t)| = 2$. Hence, energy is infinite.

Power is computed as follows.

$$\begin{aligned} \text{Power} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |2j|^2 dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T 4 dt \\ &= \lim_{T \rightarrow \infty} \frac{8T}{2T} \\ &= 4. \end{aligned}$$

Soru: Şekil fig b de verilen sinyalin toplam enerjisi nedir?



- a) $8A^2$
- b) $4A$
- c) $2A$
- d) $4A^2$

Answer: a

1.3. Gürültü

Sinyal zayıflamasının en yaygın tipi, vericide sinyal yükseltme işlemi sırasında görülen ve sinyalle birlikte yükseltme işlemine tabi tutularak yükseltilebilir gürültü çeşidi olup, çoğunlukla **termal gürültü** olarak bilinir.

Telsiz haberleşmesinde ise, termal gürültüye ilave olarak insan eliyle meydana gelen gürültü ve kanalda varolan atmosferik gürültü de alıcı antene ulaşır. Telli veya telsiz haberleşmede, kanaldaki diğer kullanıcılardan kaynaklanan gürültülerin etkisi, sinyal zayıflamasına etki eden diğer bir gürültü çeşididir. Yüksek frekanslarda yapılan telsiz haberleşmesinde, sinyal seviyesinin düşmesine sebep olan tipik bir etki de çok yönlü yayılım olup, zaman bağımlı olarak sinyal genliğinin değişimidir. Gönderilen sinyaldeki bütün bu rasgele değişim ve bozulmalar istatistik terimlerle ifade edilebilir.

Gürültü: Gönderilen orijinal sinyali bozan ve sisteme istem dışı dahil olan herhangi bir enerji (isaret ya da sinyal). Gürültünün varlığı analog veya sayısal sistemlerinin performansını düşürür. Gürültünün iletişim sisteminin performansını ne kadar etkilediği sinyal gücünün gürültü gücüne oranı (analog iletişim) ya da hata olasılığı (sayısal iletişim) ile ölçülür.

Girisim-Parazit (Interference): İstenmeyen isaretlerin sistemimize girerek sinyalimiz üzerinde yaptığı bozucu etkidir. Çözüm istenmeyen isaret kaynaklarının sistemden uzaklaştırılmasıdır.

Harmonik Bozulma: Sinyaller harmonik frekansların toplamından oluşur. Tek frekanslı bir sinüs dalgası; doğrusal olmayan bir cihazda yükseltildiğinde, bu tek frekanslı sinüs dalgasının istenmeyen katlarının oluşmasına denir. Harmonik bozulmanın çeşitli dereceleri vardır. İkinci derece harmonik bozulma v.b.

Modülasyonlar Arası Bozulma: İki veya daha çok frekans; doğrusal olmayan bir cihazda yükseltildiğinde, istenmeyen vektörel çarpımların (toplam veya fark frekanslarının) oluşmasıdır. Harmonik bozulmada olduğu gibi, bu bozulmanın da dereceleri vardır. Genel olarak 2. derece modülasyonlar arası bozulma yüzdesi kullanılır.

Çapraz Konuşma (Crosstalk): Aynı kılıf içerisinde yanyana bulunan kablolardaki sinyallerin birbirlerini etkilemeleridir. Çözüm, bükümlü kablo kullanımı)

Isıl Gürültü: Devreyi oluşturan; direnç, transistör ,vb. elemanlarda bulunan serbest elektronlar ortam sıcaklığı nedeniyle gürültü oluşturabilirler.Gürültü tarafından oluşturulan güç Johnson formülü ile ifade edilir.

$$P_n = 4kTB$$

Bu formülde;

P_n ; gürültü tarafından oluşturulan güç,

k; Boltzman sabiti $1.38 \cdot 10^{-23}$ J/K

T; sıcaklık (Kelvin)

B; bant genişliği (Hertz)

Direnç tarafından oluşturulan ısı gürültünün efektif voltaj değeri;

$e_n = \sqrt{4kTB R}$

Schottky Gürültüsü: Diyot, BJT, FET veya tüp gibi etkin bir aygıtın çıkış ögesine taşıyıcıların (elektronlar ve boşluk/delikler). Rasgele değişen bir gürültü türüdür. Var olan her işaretin üzerine binmiş olarak bulunur. Bir diyot içerisindeki schottky gürültüsü;

$I_n = \sqrt{2 \cdot q_e \cdot I_{dc} \cdot B}$

Bu formülde;

q_e ; Elektron yükü $1.6 \cdot 10^{-19}$,

I_{dc} ; Diyot doğru akımı (A)

B; Esdeğer bant genişliği (Hertz)

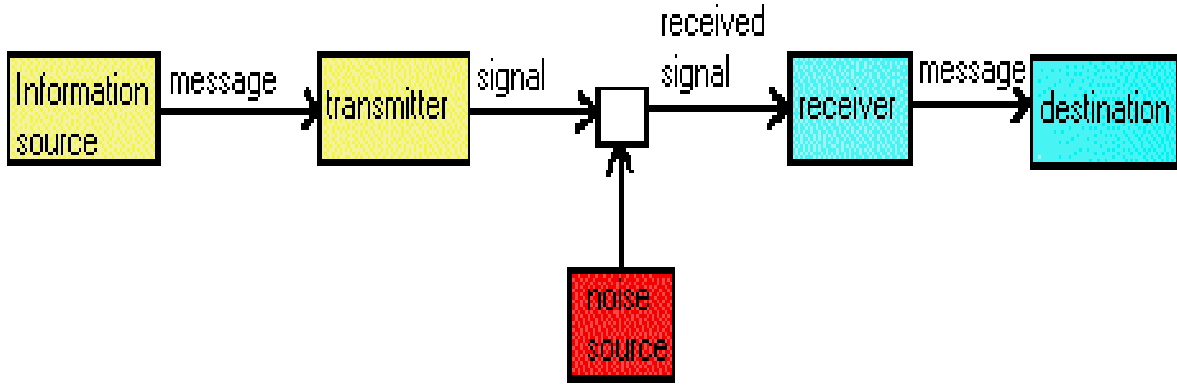
1.4. Kanal Kapasitesi

Analog sinyaller sayısal sinyallere dönüştürülürken Nyquist ve Shannon teoremleri kullanılır. Nyquist teoremi gürültüsüz iletişim ortamından gönderilebilecek toplam sayısal veri miktarını ve örnekleme aralığını belirler. Bu teoreme göre analog sinyali sayısal işarete dönüştürüp bozulmadan yeniden analog işarete dönüşebilmesi için örnekleme frekansının analog sinyalinin bant genişliğinin iki katına eşit ya da büyük olması gerekir. Örnekleme aralığı; örnekleme frekansının tersidir. Nyquist teoremi gürültüsüz iletişim kanallarını tanımlar oysa iletişim ortamları gürültü ile birlikte varlığını sürdürürler bu nedenle işaretin gürültüye oranına bağlı olarak kanal kapasitesini tanımlamak için Shannon'ın teoremi kullanılır.

Haberleşme kanalı sinyalin vericiden alıcıya gönderilmesine yarayan fiziki bir ortamdır. Elektrik sinyallerinin geçtiği, frekans ya da zaman ile ayrıştırılmış bant ya da yola kanal denir. Telsiz haberleşmesinde kullanılan kanal genellikle atmosfer (serbest uzay) olup telefon veya diğer telli haberleşme çeşitlerinde ise tel, koaksiyel kablo veya optik kablo gibi çeşitli fiziki ortamlardır. Kullanılan kanal, telli ortam da olsa gönderilen sinyal bazı rasgele zayıflama mekanizmalarına tabi olur.

Bir haberleşme sistemi tasarlanırken sinyal bozulmalarının bütün etkileri göz önüne alınmalıdır.

- Spektrum; Bir işareti oluşturan frekans aralığı; f_{min} , f_{max}
- Band Genişliği; Spektrum genişliği= $f_{max}-f_{min}$
- Kapasite; iletilebilecek data oranı bps
- Gürültü, SNR



- The maximum rate at which data can be transmitted over a given communication path
- Relationship of
 - **Data rate:** bits per second
 - **Bandwidth:** constrained by the transmitter, nature of transmission medium
 - **Noise:** depends on properties of channel
 - **Error rate:** the rate at which errors occur
- How do we make the most efficient use possible of a given bandwidth?
 - Highest data rate, with a limit on error rate for a given bandwidth

A very important consideration in data communications is how fast we can send data, in bits per second, over a channel. Data rate depends on three factors:

1. The bandwidth available
2. The level of the signals we use
3. The quality of the channel (the level of noise)

- Nyquist teoremi gürültüsüz kanal için geçerlidir
- Shannon's teoremi gürültülü kanalın kapasitesini tanımlar.

Nyquist Teoremi

Harry Nyquist (1889-1976), İsveç'te doğdu, 1907 yılında Amerika'da AT&T Bell Laboratuvarında çalıştı. Dr. Nyquist ve Dr. Claude Shannon modern telekomünikasyondaki tüm teorik gelişmelerin alt yapı teoremini geliştirmişlerdir.

Nyquist teoremi gürültüsüz haberleşme kanalından bir saniyede transfer edilecek bit miktarını tanımlar.

$$C = 2 \times B \times n$$

$$C = 64.000\text{bps}, B = 4.000\text{Hz} \text{ ise } n = ?$$

$$n = 64.000 / (2 \times 4.000) = 8 \text{ bit.}$$

Örneklenecek aralık sayısı = $2^n = 256$ dir.

Analogdan sayısala iletildikten sonra sayısaldan analoga dönüşüp orijinal işareti yeniden elde etmede kriterler;

- Örnek alma aralığı
- Örnek alınan işaretin değerin ifade edildiği sayısal bit sayısı
- Ortamın kapasitesini sınırlayan etkiler; gürültü, direnme, zayıflama

Örnek alma frekansı, $f_s \geq 2 \cdot BW$ ise (BW: band genişliği), $BW=10.000\text{Hz}$ ise örnek alma frekansını bulunuz. Örnek alma zaman aralığını saniye olarak bulunuz.

$$f_s = 2 \cdot 10.000 = 20.000\text{Hz}$$

$$T_s = 1/20000 = 50\text{mikrosaniye}$$

Soru: Analog işaret sayısal işarete dönüşüp iletildikten sonra yeniden analoga çevrildiğinde aynı işaret nasıl elde edilecektir?

Örnekleme aralığı;

300 Hz - 3400 Hz frekans aralığında işaretlerin toplamından oluşan telefon haberleşmesinde örnekleme frekansı analog işaretinin bant genişliğinin iki katı alınırsa bant genişliği bozulmaz. 125 mikro saniye aralıklarla örnek alıp, analog işaret sayısal işarete dönüştürülürse orijinalite bozulmaz.

Eğer bir $f(t)$ analog işareti, bant genişliğinin 2 katına eşit ya da büyük bir frekansla eşit aralıklarla örneklenirse sayısal işareten orijinal analog işaret yeniden elde edilir. Analog işareti sayısala çevirdikten sonra: Transfer edilir, bilgisayarda işlenir, tekrar duyabilmek için analoga çevrilir. Nyquist'e göre: Analog frekansım, bant genişliğinin 2 katından büyük ya da eşit olmalıdır.

$f_s \geq 2 \cdot B_w$, telefon iletişim teknolojisinde 0 – 4 kHz arasındadır.

$$T_s = \frac{1}{f_s} \quad f_s \geq 2 \cdot 4\text{KHz} \geq 8\text{KHz}$$

$$T_s = \frac{1}{f_s} = \frac{1}{8000} = 125 \cdot 10^{-6} \text{ sec} = 125 \mu\text{sec}$$

Analog işareten 125 mikro saniyede bir örnek alınıp transfer edilip, tekrar analoga çevrilirse işaretin orijinalitesi bozulmaz.

- Müzikte bant genişliği (Bw) 10 ile 12 kHz arasındadır.
- Çok yüksek kalite müziklerde Bw, 20kHz olmalıdır.

$$\text{MaxDataRate} = 2B \log_2 V \quad [\text{bits/sec}]$$

$$C = 2 \cdot B \cdot \log_2 (2^n)$$

C= Kanaldan gönderilebilecek 1 saniyedeki bit sayısı (bps)

B = Kanal Band genişliği (Hz)

$V=2^n$ = Örnek alınacak analog işaretin bölüneceği seviye sayısı

Shannon Teoremi (Claude Shannon 1916-2001)

$$MaxDataRate = B \log_2 \left(1 + \frac{S}{N} \right) \quad [bits/sec]$$

$$C = W \log_2 \left(1 + \frac{S}{N} \right)$$

$$\begin{cases} S = E_b C \\ N = N_0 W \end{cases}$$

$$\frac{C}{W} = \log_2 \left(1 + \frac{E_b}{N_0} \frac{C}{W} \right)$$

As $W \rightarrow \infty$ or $\frac{C}{W} \rightarrow 0$, we get :

$$\frac{E_b}{N_0} \rightarrow \frac{1}{\log_2 e} = 0.693 \approx -1.6 \text{ [dB]}$$

Shannon limit:

- **There exists a limiting value of below which there can be no error-free communication at any information rate.**
- **By increasing the bandwidth alone, the capacity can not be increased to any desired value.**

İşaretin Gürültüye Oranı

SNR = 10 Log (S/N)

S: İşaret Seviyesi (Watt)

N: Gürültü Seviyesi (Watt)

Shannon Kapasite Formülü:

Veri hızını iletim gürültüsü ile ilişkilendirir.

İşaretin gücünün yüksek olması hatayı azaltacaktır.

$$C = B \log_2 (1 + SNR)$$

Band genişliğini artırmak kapasiteyi artırır, ancak SNR oranını azaltır.

$SNR = \frac{E_b}{N_0}$ olarak verilir. Burada E_b bit enerjisi, N_0 ise Hertz başına gürültü gücüdür.

$$64000 = B \log_2 \left(1 + \frac{S}{N}\right)$$

$$64000 = 4000 \log_2 \left(1 + \frac{S}{N}\right)$$

$$16 = \log_2 \left(1 + \frac{S}{N}\right)$$

$$16 = \log_2 \left(1 + \frac{S}{N}\right)$$

$$\left(1 + \frac{S}{N}\right) = 2^{16}$$

$$\frac{S}{N} = 2^{16} - 1$$

$$10 \log \left(\frac{S}{N}\right) = 10 \log 2^{16}$$

$$40$$

$$\frac{S}{N} = 10^{3.8}$$

$$10 \log \frac{10}{1} = 10 \log 10 = 10 \text{ dB}$$

$$\frac{S}{N} = 10 \text{ dB}$$

$$\left(\frac{S}{N}\right)_{\text{dB}} = 10 \log \left(\frac{S}{N}\right)$$

$$10 \log \left(\frac{S}{N}\right) = 10 \log 10^3 = 30 \log 10 = 30 \text{ dB}$$

$$\log_2 2^n = ?$$

$$\frac{64000}{4000} = 16$$

$$1 + \frac{S}{N} = 4$$

$$\frac{S}{N} = 3$$

$$\log_2 1001 = ?$$

$$\log_2 2^{10} = 10$$

$$B_w: 4 \text{ kHz}$$

$$F_s \geq 2 B_w = 8 \text{ kHz}$$

$$T_s = \frac{1}{8 \text{ kHz}} = 125 \mu\text{sec}$$

$$C = 2 \times 4000 \times 8 = 64.000 \text{ bit/sec}$$

$$\frac{S}{N} = 10^3$$

Soru:

Zamanla genliği, frekansı ve fazı değişen sinüsoidal işaretlerin birleşimden analog işaret elde edilmektedir. Analog işareti sayısal işarete dönüştürürken Nyquist teoremi kullanılır. Bu teoreme göre örnekleme frekansı maksimum frekansın iki katı seçilir. Nyquist oranı, örtüşmeyi (aliasing) önlemek için minimum örnekleme sıklığıdır.

Aşağıda frekansları verilen üç adet sinusoidal sinyalden oluşan analog sinyalin Nyquist oranına göre örnekleme frekansını bulunuz.

$$F_1 = 1000 \text{ Hz}, F_2 = 3000 \text{ Hz} \text{ and } F_3 = 6000 \text{ Hz}.$$

Kanal kapasitesi maksimum veri akış hızıdır. Veri hızı, band genişliği, gürültü, hata oranı ile ilişkilidir. Veri hızı saniyedeki veri işleme ya da iletilen bit sayısıdır.

Nyquist Bant Genişliği:

- Kanal band genişliği veri hızına bağlıdır.
- B band genişlikli bir kanaldan 2B bit hızıyla haberleşilir.
- Çok seviyeli sinyaller varsa Band kapasitesi, $C = 2B \log_2 M$ olacaktır.

Rayleigh Dağılımı:

Hareketli haberleşme kanallarında Rayleigh dağılımı düz sönümlenmeye uğramış işaretin alıcıdaki zarfının istatistiksel olarak zamanla değişimini gösterir. Gürültü işaretinin dik bileşenleri Gauss dağılımı şeklinde ifade edilip zarfı Rayleigh dağılımını vermektedir. Rayleigh dağılımının olasılık yoğunluk fonksiyonu aşağıdaki gibi ifade edilir.

Gezgin iletisimde alıcı ve verici arasında direkt görüs yolu (line-of-sight, LOS) yoksa Rayleigh dağılımıyla modellenir. Rayleigh dağılımı, düz sönülmeye uğramıs ve alıcıya doğrudan ulasmayan isaretin alıcıdaki zarfının istatistiksel olarak zamanla deęişimini tanımlamaktadır. Rayleigh dağılımı ařaęıdaki olasılık yoęunluk islevine sahiptir:

$$P_{Rayleigh}(r) = \begin{cases} \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) & 0 \leq r \leq \infty \\ 0 & r < 0. \end{cases}$$

Rayleigh dağılımının ortalama deęeri,

$$r_{mean} = E[r] = \int_0^{\infty} rp(r)dr = \sigma\sqrt{\frac{\pi}{2}} = 1.2533\sigma$$

olarak bulunur. Rayleigh dağılımının varyansı ise

$$\begin{aligned} \sigma_r^2 &= E[r^2] - E^2[r] = \int_0^{\infty} r^2 p(r)dr - \frac{\sigma^2\pi}{2} \\ &= \sigma^2\left(2 - \frac{\pi}{2}\right) = 0.4292\sigma^2 \end{aligned}$$

řeklinde bulunur.

Rician dağılımı ařaęıdaki gibidir:

$$P_{Rician}(r) = \frac{r}{\sigma^2} e^{-\frac{(r^2+A^2)}{2\sigma^2}} I_0\left(\frac{rA}{\sigma^2}\right), \quad A \geq 0 \text{ ve } r \geq 0.$$

Gecikme

Gecikme çeşitleri:

- İşleme
- Protokol Kod Çalıştırmasına Zamanı
- Kuyruk
- İşlenmek Üzere Kuyrukta Bekleme Süresi
- Transmisyon
- "Bit Ortamdan Gönderme Süresi"
- Yayılma

Yayılm ve iletim gecikmesi

Propagation speed - speed at which a bit travels though the medium from source to destination.

Transmission speed - the speed at which all the bits in a message arrive at the destination. (difference in arrival time of first and last bit)

Propagation Delay = Distance/Propagation speed

Transmission Delay = Message size/bandwidth bps

Latency = Propagation delay + Transmission delay + Queueing time + Processing time

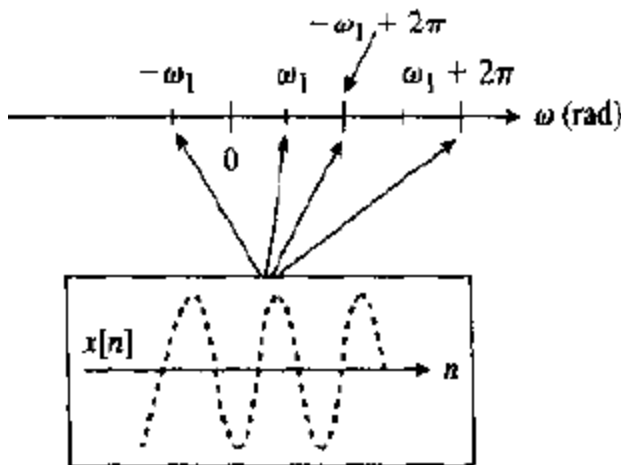
1.5. Örnekleme

Sampling continuous-time sinusoid and aliasing

If two sequences $x_1(n) = A\cos(\omega_1 n + \alpha_1)$ and $x_2(n) = A\cos(\omega_2 n + \alpha_2)$ have frequencies and phases related by,

$$\omega_2 = \omega_1 + 2k\pi, \quad \alpha_2 = \alpha_1 \quad \text{or,} \quad \omega_2 = -\omega_1 + 2k\pi, \quad \alpha_2 = -\alpha_1$$

with k an integer, then the two sinusoidal sequences have the same samples, i.e. $x_1(n) = x_2(n)$. This is illustrated in Figure below.



Here, $\omega_1, \omega_1 + 2\pi, -\omega_1$ and $-\omega_1 + 2\pi$ represents the same signal in the time domain. If we limit the digital frequency ω within the interval $-\pi$ to π then there is one to one correspondence between the signals and their frequency representation. For each frequency in the interval $-\pi$ to π the corresponding aliases are all outside the interval $-\pi$ to π itself.

Now, the range of unique digital frequencies,

$$-\pi \leq \omega \leq \pi \Rightarrow -\pi \leq \Omega T \leq \pi$$

$$\text{Or, } -\pi/T \leq \Omega \leq \pi/T \text{ or, } -\pi F_s \leq \Omega \leq \pi F_s$$

$$\text{Or, } -\frac{\Omega_s}{2} \leq \Omega \leq \frac{\Omega_s}{2}$$

This implies that the highest frequency of an analog signal must be less than half the sampling frequency to avoid aliasing.

Example:

Consider the analog signal $x_a(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12000\pi t$. What is the Nyquist rate for this signal? If the sampling rate is 5000 samples/sec what is the discrete-time signal obtained after sampling? What is the analog signal $y_a(t)$ we can reconstruct from the samples if we use ideal interpolation?

Nyquist rate is the minimum sampling frequency to avoid aliasing. This is double the maximum frequency of input signal.

$$\text{Here, } F_1 = 1000 \text{ Hz, } F_2 = 3000 \text{ Hz and } F_3 = 6000 \text{ Hz.}$$

$$\text{Thus, Nyquist rate, } F_N = 12000 \text{ Hz.}$$

$$\begin{aligned} x(n) = x_a(nT_s) &= 3\cos \frac{2000\pi}{5000}n + 5\sin \frac{6000\pi}{5000}n + 10\cos \frac{12000\pi}{5000}n \\ &= 3\cos \frac{2\pi}{5}n + 5\sin \frac{6\pi}{5}n + 10\cos \frac{12\pi}{5}n \end{aligned}$$

$$\text{Or, } x(n) = 3\cos \frac{2\pi}{5}n + 5\sin(2\pi - \frac{4\pi}{5})n + 10\cos(2\pi + \frac{2\pi}{5}n)$$

$$= 3\cos \frac{2\pi}{5}n - 5\sin \frac{4\pi}{5}n + 10\cos \frac{2\pi}{5}n$$

$$\text{Or, } x(n) = 13\cos \frac{2\pi}{5}n - 5\sin \frac{4\pi}{5}n. \quad \therefore f_1 = \frac{1}{5}, f_2 = \frac{2}{5}$$

$$\text{For perfect reconstruction, } F_1 = f_1 \cdot F_s = \frac{1}{5} \cdot 5000 \text{ Hz} = 1000 \text{ Hz,} \quad \text{and}$$

$$\therefore F_2 = f_2 \cdot F_s = \frac{2}{5} \cdot 5000 \text{ Hz} = 2000 \text{ Hz.}$$

$$\text{Thus, } y_a(t) = 13\cos 2000\pi t - 5\sin 4000\pi t.$$

Note that, F_1 is less than 2500 Hz. So no aliasing will occur. F_2 is greater than 2500 Hz by 500 Hz. So this frequency will appear as a lower frequency of $(2500 - 500) = 2000$ Hz. Alternately, $(3000 - 5000) = -2000$ Hz. The third frequency will change to $(6000 - 5000) = 1000$ Hz.

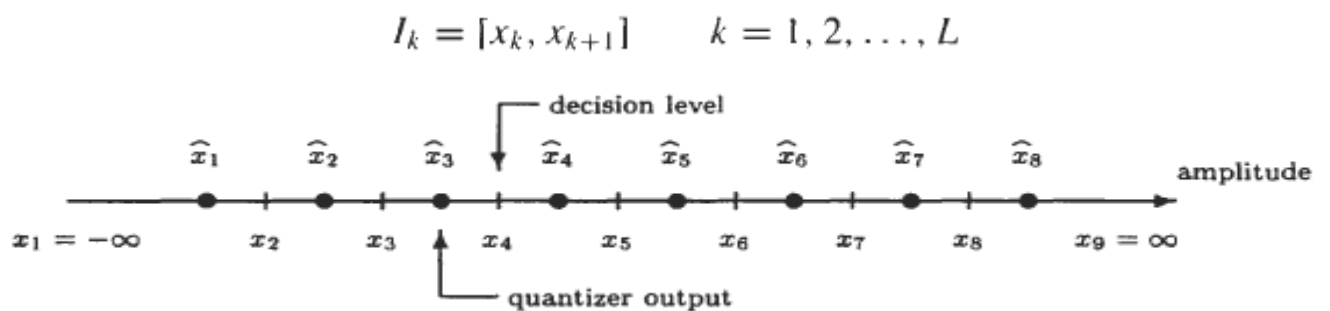
Quantization and Encoding

Quantization is a process that converts data from infinite or high precision to finite or lower precision. The error introduced in representing the continuous-valued signal by a finite set of discrete value levels is called quantization error and is denoted by $e(n)$.

$$\begin{cases} \hat{x}(n) = Q[x(n)] \\ e(n) = \hat{x}(n) - x(n) \end{cases}$$

The distance Δ between two successive quantization levels is called **quantization step size** or **resolution**.

The quantizer has $(L+1)$ decision levels x_1, x_2, \dots, x_{L+1} that divide the amplitude range for $x(n)$ into L intervals.



For an input $x(n)$ that falls in the interval I_k , the quantizer assigns a value within the interval $\hat{x}(k)$ to $x(n)$. This process is shown in Figure above.

$$\Delta = x_{k+1} - x_k.$$

The number of levels in a quantizer is generally of the form, $L = 2^{B+1}$ for a $B+1$ bit binary code word.

A 3-bit uniform quantizer in which the quantizer output is rounded to the nearest quantization level is shown in Figure below.

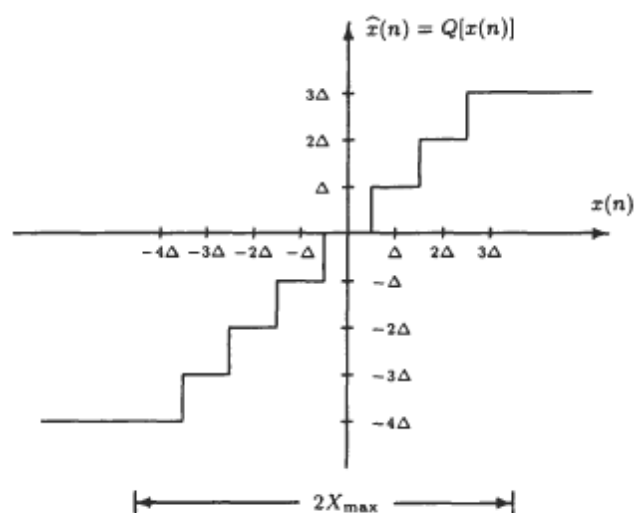


Fig. 3-4. A 3-bit uniform quantizer.

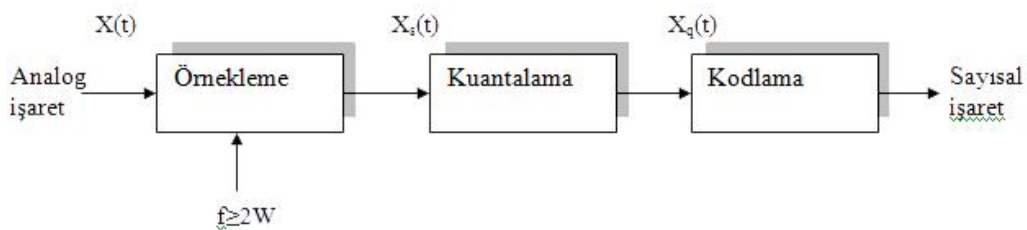
With L quantization levels, the range of the quantizer is, $R = 2^{B+1} \cdot \Delta$. If the quantizer input is bounded, $|x(n)| \leq X_{\max}$.

Darbe kod modülasyonu (PCM), PCM'de, darbeler sabit uzunlukta ve sabit genliktedir. PCM ikili bir sistemdir; önceden belirlenmiş bir zaman bölümü içinde bir darbenin bulunması ya da bulunmaması, 1 ya da 0 mantık durumunu gösterir. PWM, PPM ve PAM'da, tek bir ikili sayıyı (bit) göstermez.

Alma ucunda, sayısal / analog dönüştürücü (DAC), seri ikili veri akışını çok düzeyli bir PAM sinyale dönüştürür. Örnekleme ve tutma devresi ile alçak geçiren filtre, PAM sinyali tekrar başlangıçtaki analog biçimine dönüştürür. PCM kodlamayı ve kod çözümü gerçekleştiren entegre devreye kodek (kodlayıcı / kod çözücü) denir.

Darbe modülasyonunda, analog enformasyonun ayrık zamanda iletişimi söz konusudur. PAM, PWM ve PPM modülasyonlarıyla darbenin sırasıyla genliğinin, genişliğinin ve bir periyot içindeki pozisyonunun sürekli olarak tüm işaret değerleri için değişimine izin verilmektedir.

Zamanda ayrık duruma getirilmiş (örneklenmiş) işaretin genliğinin de belirli sayıda ayrık seviyelere ayrılarak kuantalanmasıdır (kuantalama; belirli örnekleme zamanlarında elde edilen genlik numuneleri). Örneklenmiş sürekli genlikli işareti, belirli seviyelere kuantalamakla kalmayıp bir örnek anında her seviye için bir kod kullanılacaktır. Bu türden modülasyon darbe kod modülasyonu (pulse code modulation – PCM) olarak adlandırılır. PCM'de enformasyon taşıyan $x(t)$, işareti önce uygun bir örnekleme frekansı ile örneklenir. Daha sonra bu örnek değerler, belirli kuantalama seviyelerine kuantalanır. Buna kuantalama işlemi adı verilir. Son olarak, her kuantalama seviyesi bir ikili kod kelimesi ile, yani sonlu sayıda (0,1) dizisi ile gösterilir. İkili kod kelimeler dizisine dönüştürülen bu işarete PCM dalgası adı verilir. PCM sistemi verici bölümünün blok diyagramı görülmektedir.



PCM'de verici bölümün blok diyagramı

Örnekleme Devresi

Nyquist örnekleme teoremi, bir PCM sistem için kullanılacak minimum örnekleme hızını (f_s) belirler. Bir örneklemin alıcıda doğru olarak tekrar oluşturulabilmesi için, analog giriş sinyalinin (f_a) her çevrimi en az iki kez örneklenmelidir. Dolayısıyla, minimum örnekleme hızı, en yüksek ses giriş frekansının iki katına eşittir. f_s , f_a 'nın iki katından daha küçükse, bozulma meydana gelir. Bu bozulmaya katlama bozulması denir. Minimum Nyquist örnekleme hızı, matematiksel olarak, $f_s \geq 2f_a$ ifade edilir. Burada; f_s = Minimum Nyquist örnekleme hızı, f_a = Örneklenebilecek en yüksek frekans

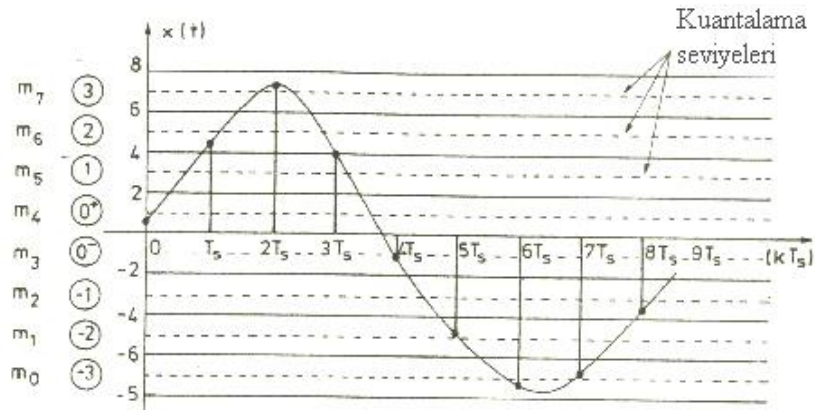
Kuantalama İşlemi

Darbe genlik modülasyonunda, örneklenmiş değerler belirli kuantalama seviyelerine yuvarlatılmadan iletilmektedir. Ancak, bu işlem işaretin gürültüye olan bağışıklığı açısından bir yarar sağlamayacaktır. Bunun yerine, işaret genliğini belirli kuantalama seviyelerine yuvarlatmak ve her kuantalama seviyesi için, uygun bir kod kelimesi kullanmak daha uygun olmaktadır. $x(t)$ işaretinin maksimum ve minimum genlikleri A_{max} ile $-A_{max}$ arasında değişiyorsa ve bu aralıkta değişen genlik değerleri $Q = 2^n$ adet eşit kuantalama seviyesine bölünmek isteniyorsa kuantalama aralığı veya adımı;

$$a = 2A_{max} / 2^n$$

olarak tanımlanmaktadır. Kuantalama işleminde örnek değerlerin bulunduğu dilim belirlenir.

Çeşitli işaret genliklerine karşılık gelen kuantalama seviyeleri ve kod kelimeleri

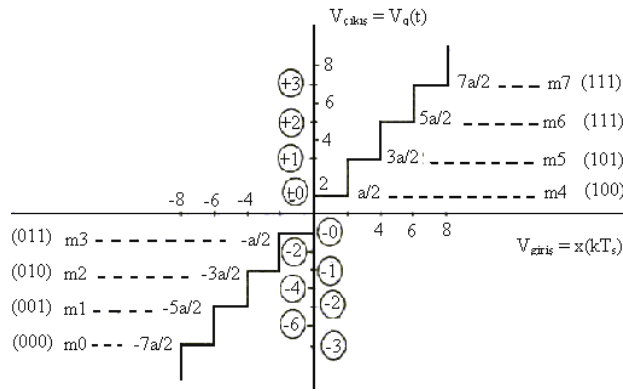


Kuanta seviyeleri

Kuantalama dilim sayısı Q arttıkça, kuantalama gürültüsü de azalacaktır. Buna karşılık bir örneği belirlemek için kullanılması gerekli bit sayısı da artacaktır.

PCM sistemindeki kuantalama hatası, kuvvetli işaretlerde ihmal edilebilecek kadar küçük olmasına rağmen, zayıf işaretlerde kuantalama seviyesi ne olursa olsun önemlidir. Bu hatayı önlemek amacı ile verici tarafta; sıkıştırma ve alıcı tarafta; genişletme işlemleri yapılmaktadır. Sıkıştırma işlemi ile büyük genlikler zayıflatılarak küçük genliklerin seviyesine düşürülür. Bu teknik PCM ve delta modülasyon tekniklerinin temelini oluşturur.

Lineer Kuantalama



Düzgün kuantalama eğrisi

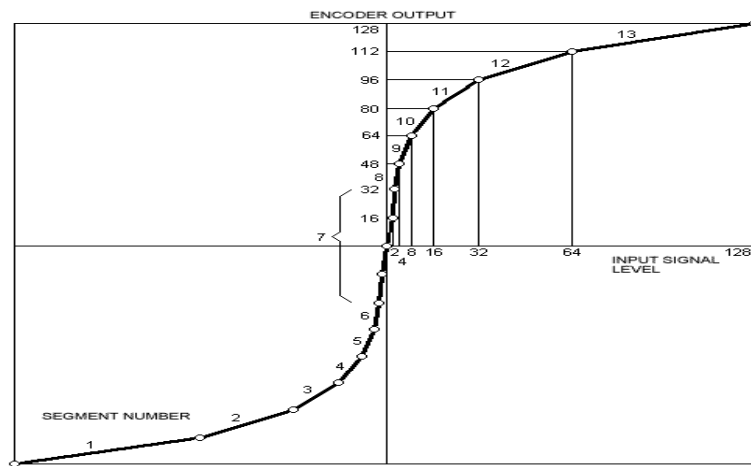
Lineer Olmayan Kuantalama

Ses işaretlerinin istatistikleri incelendiğinde, küçük genliklere daha sık rastlanıldığı görülmektedir. Oysa, yukarıda küçük işaretlerde kuantalama gürültüsünün rahatsız edici boyutlarda olacağı gösterilmiş bulunmaktadır.

Bu gürültüyü azaltmak için başvurulacak ilk yöntem, adım büyüklüğünün azaltılması veya dilim sayısının artırılmasıdır. Ancak, bu durumda her bir örneği göstermek için kullanılması gereken bit sayısı artacağından, bu yöntem her zaman uygun ve ekonomik değildir. Diğer taraftan, çok seyrek olarak ortaya çıkan yüksek genlikli işaretler için gereksiz yere bir miktar dilim ayrılmış olacaktır. Eğer en büyük genlik küçük tutulursa, bu defa da kırılmalar meydana gelecektir.

Örneğin, dinamiği 36 dB olan bir ses işareti ele alınırsa, en küçük işaretle 36 dB işaret gürültü oranı elde etmek için, $n = 12$ bitlik kelime uzunluğunda bir PCM kullanmak gerekecektir. Böyle bir kuantalayıcı da, en küçük işaret için 36 dB işaret gürültü oranı sağlanırken, en büyük işaret için gereksiz yere 72 dB'lik bir işaret gürültü oranı elde edilecektir. Bununla beraber, büyük işaretler için büyük adım, küçük işaretler için de küçük adım kullanılarak, işaret gürültü oranının aynı olması sağlanabilir. Bunu gerçekleştirebilmek için, haberleşme sistemlerinde bir sıkıştırma (compressing) yapılmaktadır.

In the following image is represented the A law (a-law) graphically



Kodlama İşlemi

Örneklenmiş analog işareti kuantaladıktan sonra, sayısal (dijital) işarete dönüştürmek için kodlanması gerekmektedir. Genellikle bir analog işareti sayısal (dijital) işarete dönüştürmek için, analog / sayısal dönüştürücüler (ADC) kullanılır. Bu dönüştürme işlemi sonucunda kodlanmış işaret elde edilmektedir. Ancak bu kodlama, düzgün (uniform) bir kuantalama sonucu elde edilen genliklerin kodlanması olup, düzgün olmayan kuantalama sonucu bulunan örneklerin kodlanması için ikinci bir kodlama işlemine ihtiyaç olacaktır. Uygulamada kullanılan analog / sayısal dönüştürücü (ADC) tiplerini üç gruba ayırmak mümkündür.

- Basamaklı dönüştürücü
- Ardışıl yaklaşımlı dönüştürücü
- Paralel dönüştürücü

Değişik tekniklerde çalışan bu analog /sayısal dönüştürücüleri, tüm devreler biçiminde piyasada bulmak mümkündür. Bu tip tüm devrelerde genellikle, örnekleme, kuantalama ve kodlama birlikte gerçekleştirilir.

1.6. Ayrik Zamanlı Sinyaller

The discrete-time signals are used as basis functions or building blocks to describe more complex signals. A discrete-time signal $x(n)$ is a function of an independent variable that is an integer. It is graphically represented as in Fig. 2.1. It is important to note that a discrete-time signal is *not* defined at instants between two successive samples.

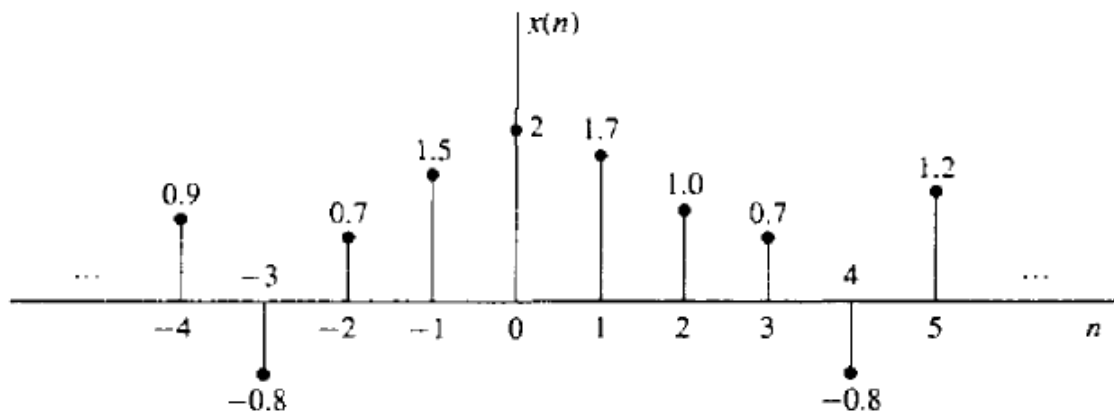


Figure :Graphical representation of a discrete-time signal.

In the sequel we will assume that a discrete-time signal is defined for every integer value n for $-\infty < n < \infty$. By tradition, we refer to $x(n)$ as the "nth sample" of the signal even if the signal $x(n)$ is inherently discrete time (i.e., not obtained by sampling an analog signal). If, indeed, $x(n)$ was obtained from sampling an analog signal $x_a(t)$, then $x(n) = x_a(nT)$, where T is the sampling period (i.e., the time between successive samples).

Besides the graphical representation of a discrete-time signal or sequence as illustrated in Fig. 2.1, there are some alternative representations that are often more convenient to use. These are:

1. Functional representation, such as

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases}$$

2. Tabular representation, such as

n	...	-2	-1	0	1	2	3	4	5	...
$x(n)$...	0	0	0	1	4	1	0	0	...

3. Sequence representation

An infinite-duration signal or sequence with the time origin ($n = 0$) indicated by the symbol f is represented as

$$x(n) = \{ \dots 0, 0, 1, 4, 1, 0, 0, \dots \}$$

A sequence $x(n)$, which is zero for $n < 0$, can be represented as

$$x(n) = \{ 0, 1, 4, 1, 0, 0, \dots \}$$

A finite-duration sequence can be represented as

$$x(n) = \{ 3, -1, -2.5, 0, 4, -1 \}$$

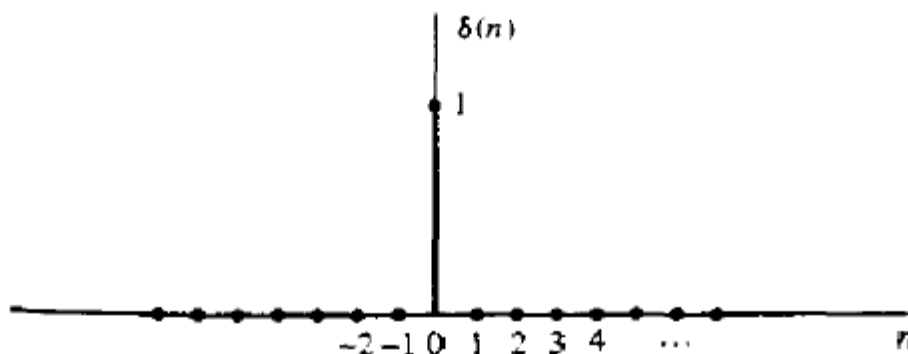
Some Elementary Discrete-Time Signals

In our study of discrete-time signals and systems there are a number of basic signals that appear often and play an important role. These signals are defined below.

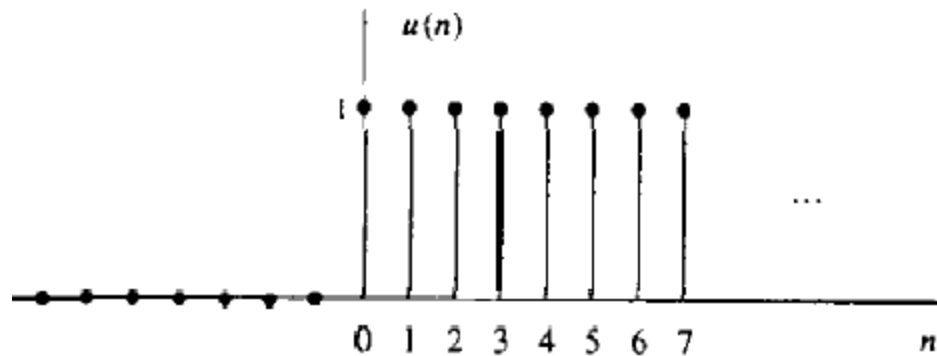
1. The **unit sample sequence**: is denoted as $\delta(n)$ and is defined as

$$\delta(n) \equiv \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases}$$

Note: Unit Impulse in analog signals

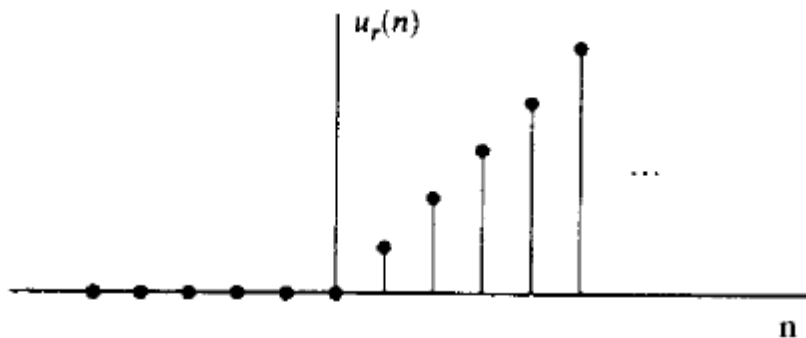


2. The *unit step signal* is denoted as $u(n)$ and is defined as



3. The *unit ramp signal* is denoted as $u_r(n)$ and is defined as

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$



4. The *exponential signal* is a sequence of the form

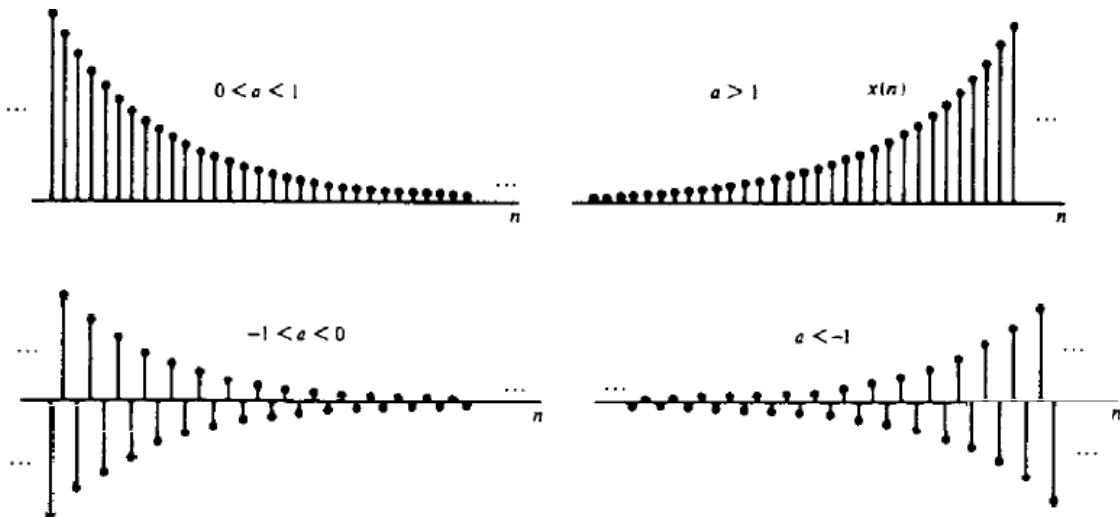
$$x(n) = a^n \quad \text{for all } n$$

If the parameter a is real, then $x(n)$ is a real signal. As shown below for various values of the parameter a . When the parameter a is complex valued, it can be expressed as

$$a \equiv re^{j\theta}$$

where r and θ are now the parameters. Hence we can express $x(n)$ as

$$\begin{aligned} x(n) &= r^n e^{j\theta n} \\ &= r^n (\cos \theta n + j \sin \theta n) \end{aligned}$$



Since $X(n]$ is now complex valued, it can be represented graphically by plotting the real part

$$x_R(n) \equiv r^n \cos \theta n$$

as a function of n , and separately plotting the imaginary part

$$x_I(n) \equiv r^n \sin \theta n$$

as a function of n .

Ayrık zamanlı sinyallerin sınıflandırılması

The mathematical methods employed in the analysis of discrete-time signals and systems depend on the characteristics of the signals. We classify discrete-time signals according to a number of different characteristics.

Energy signals and power signals. The energy E of a signal $x(n)$ is defined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2$$

The energy of a signal can be finite or infinite. If E is finite (i.e., $0 < E < \infty$), then $x(n)$ is called an energy signal. Many signals that possess infinite energy, have a finite average power. The average power of a discrete-time signal $x(n)$ is defined as

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x(n)|^2$$

If we define the signal energy of $x(n)$ over the finite interval $-N \leq n \leq N$ as

$$E_N \equiv \sum_{n=-N}^N |x(n)|^2$$

then we can express the signal energy E as

$$E \equiv \lim_{N \rightarrow \infty} E_N$$

and the average power of the signal $x(n)$ as

$$P \equiv \lim_{N \rightarrow \infty} \frac{1}{2N+1} E_N$$

Clearly, if E is finite, $P = 0$. On the other hand, if E is infinite, the average power P may be either finite or infinite. If P is finite (and nonzero), the signal is called a power **signal**. The following example illustrates such a signal.

Example .

Determine the power and energy of the unit step sequence. The average power of the unit step signal is

$$\begin{aligned}
 P &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=0}^N u^2(n) \\
 &= \lim_{N \rightarrow \infty} \frac{N+1}{2N+1} = \lim_{N \rightarrow \infty} \frac{1+1/N}{2+1/N} = \frac{1}{2}
 \end{aligned}$$

Consequently, the unit step sequence is a power signal. Its energy is infinite. Similarly, it can be shown that the complex exponential sequence $x(n) = Ae^{j\omega n}$ has average power A^2 , so it is a power signal. On the other hand, the unit ramp sequence is neither a power signal nor an energy signal.

Periodic signals and aperiodic signals. A signal $x(n)$ is periodic with period N ($N > 0$) if and only if

$$x(n + N) = x(n) \text{ for all } n$$

The smallest value of N is called the (fundamental) period. If there is no value of N that satisfies equation above, the signal is called *nonperiodic* or **aperiodic**.

We have already observed that the sinusoidal signal of the form

$$x(n) = A \sin 2\pi f_0 n$$

is periodic when f_0 is a rational number, that is, if f_0 can be expressed as

$$f_0 = \frac{k}{N}$$

where k and N are integers.

The energy of a periodic signal $x(n)$ over a single period, say, over the interval $0 \leq n \leq N-1$, is finite if $x(n)$ takes on finite values over the period. However, the energy of the periodic signal for $-\infty \leq n \leq \infty$ is infinite. On the other hand, the average power of the periodic signal is finite and it is equal to the average power over a single period. Thus if $x(n)$ is a periodic signal with fundamental period N and takes on finite values, its power is given by

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

Consequently, periodic signals are power signals.

Symmetric (even) and antisymmetric (odd) signals. A real-valued signal $x(n)$ is called symmetric (even) if

$$x(-n) = x(n)$$

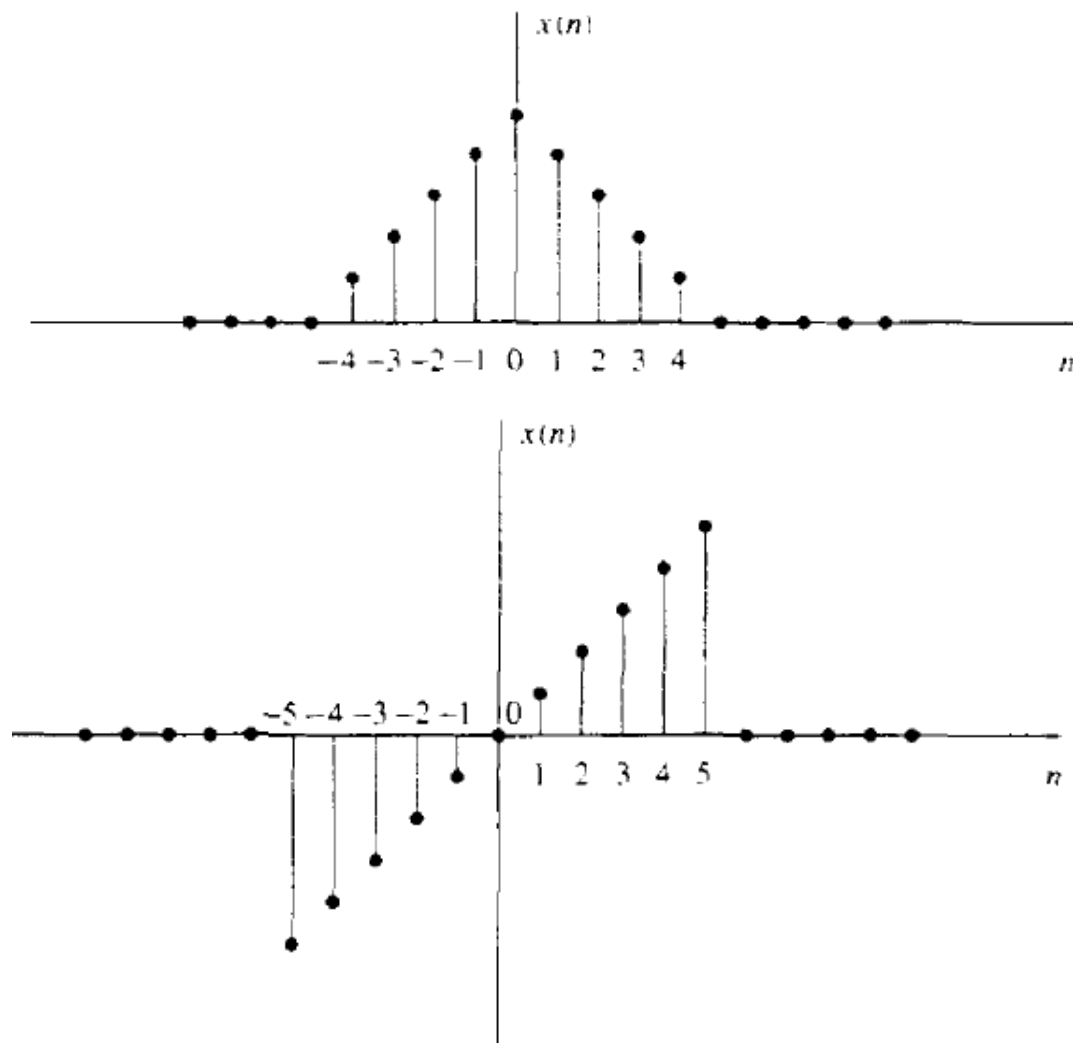
On the other hand, a signal $x(n)$ is called antisymmetric (odd) if

$$x(-n) = -x(n)$$

We note that if $x(n)$ is odd, then $x(0) = 0$. Examples of signals with even and odd symmetry are illustrated in Fig below.

We wish to illustrate that any arbitrary signal can be expressed as the sum of two signal components. one of which is even and the other odd. The even signal component is formed by adding $x(n)$ to $x(-n)$ and dividing by 2. that is.

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$



Similarly, we form an odd signal component $x_o(n)$ according to the relation

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

Now, if we add the two signal components, we obtain $x(n)$, that is,

Solved Problems

$$x(n) = x_e(n) + x_o(n)$$

1-Find the even and odd parts of the following signals:

(a) $x(n) = u(n)$

(b) $x(n) = \alpha^n u(n)$

The even part of a signal $x(n)$ is given by

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

Any arbitrary signal can be expressed as in equation above.

With $x(n) = u(n)$, we have

$$x_e(n) = \frac{1}{2}[u(n) + u(-n)] = \begin{cases} 1 & n = 0 \\ \frac{1}{2} & n \neq 0 \end{cases}$$

which may be written concisely as

$$x_e(n) = \frac{1}{2} + \frac{1}{2}\delta(n)$$

Therefore, the even part of the unit step is a sequence that has a constant value of 1/2 for all n except at $n = 0$, where it has a value of 1.

The odd part of a signal $x(n)$ is given by the difference

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

With $x(n) = u(n)$, this becomes

$$x_o(n) = \begin{cases} \frac{1}{2} & n > 0 \\ 0 & n = 0 \\ -\frac{1}{2} & n < 0 \end{cases}$$

or

$$x_o(n) = \frac{1}{2}\text{sgn}(n)$$

where $\text{sgn}(n)$ is the signum function.

With $x(n) = \alpha^n u(n)$, the even part is

$$x_e(n) = \frac{1}{2}[\alpha^n u(n) + \alpha^{-n} u(-n)] = \begin{cases} \frac{1}{2}\alpha^n & n > 0 \\ 1 & n = 0 \\ \frac{1}{2}\alpha^{-n} & n < 0 \end{cases}$$

or

$$x_e(n) = \frac{1}{2}\alpha^{|n|} + \frac{1}{2}\delta(n)$$

The odd part, on the other hand, is

$$x_o(n) = \frac{1}{2}[\alpha^n u(n) - \alpha^{-n} u(-n)] = \frac{1}{2}\alpha^{|n|} \text{sgn}(n)$$

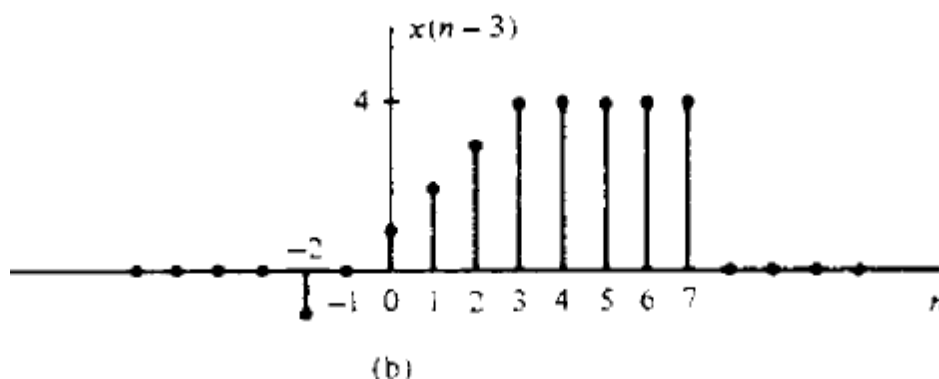
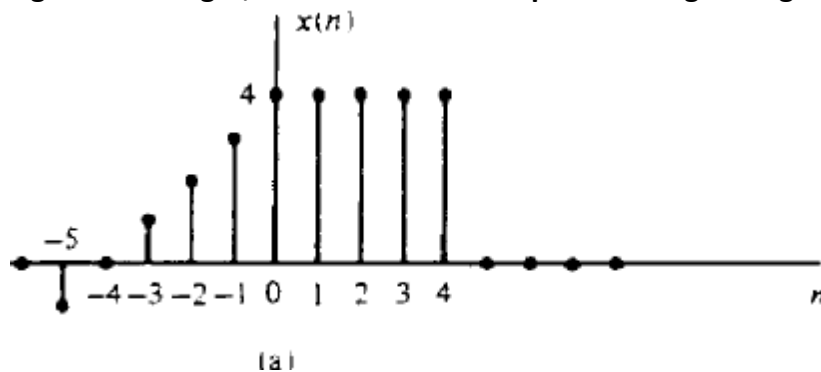
Simple Manipulations of Discrete-Time Signals

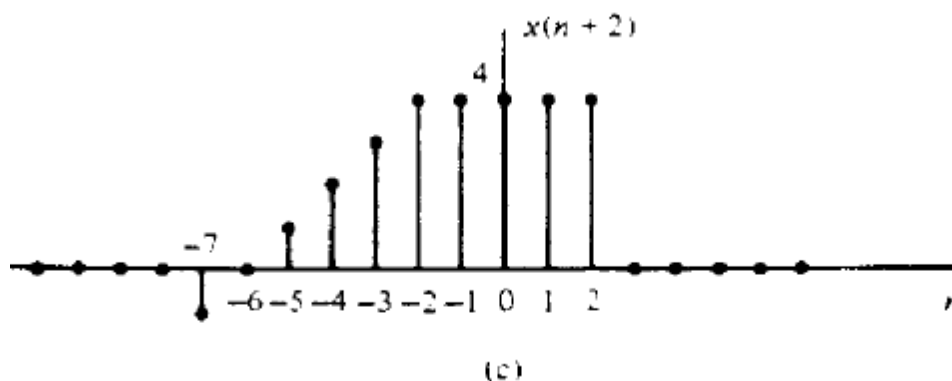
We consider some simple modifications or manipulations involving the independent variable and the signal amplitude (dependent variable).

Transformation of the independent variable (time). A signal $x(n)$ may be shifted in time by replacing the independent variable n by $n - k$, where k is an integer. If k is a positive integer, the time shift results in a delay of the signal by k units of time. If k is a negative integer, the time shift results in an advance of the signal by $|k|$ units in time.

Example A signal $x(n)$ is graphically illustrated in Fig.a. Show a graphical representation of the signals $x(n - 3)$ and $x(n + 2)$.

Solution: The signal $x(n - 3)$ is obtained by delaying $x(n)$ by three units in time. The result is illustrated in b. On the other hand, the signal $x(n + 2)$ is obtained by advancing $x(n)$ by two units in time. The result is illustrated in Fig c. that delay corresponds. Note shifting a signal to the right, whereas advance implies shifting the signal to the left on the time axis.





If the signal $x(n)$ is stored on magnetic tape or on a disk or, perhaps, in the memory of a computer, it is a relatively simple operation to modify the base by introducing a delay or an advance. On the other hand, if the signal is not stored but is being generated by some physical phenomenon in real time, it is not possible to advance the signal in time, since such an operation involves signal samples that have not yet been generated. Whereas it is always possible to insert a delay into signal samples that have already been generated, it is physically impossible to view the future signal samples. Consequently, in real-time signal processing applications, the operation of advancing the time base of the signal is physically unrealizable.

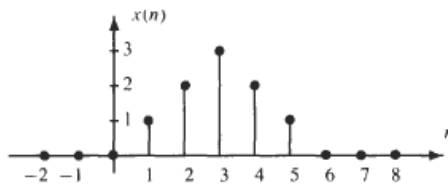
Another useful modification of the time base is to replace the independent variable n by $-n$. The result of this operation is a **Folding** or a **Reflection Or Flipping or Reversal** of the signal about the time origin $n = 0$.

A third modification of the independent variable is Time Scaling. This transformation is defined by $f(n) = Mn$ or $f(n) = n/N$ where M and N are positive integers. In the case of $f(n) = Mn$, the sequence $x(Mn)$ is formed by taking every M th sample of $x(n)$ (this operation is known as down-sampling). With $f(n) = n/N$ the sequence $y(n) = x(f(n))$ is defined as follows:

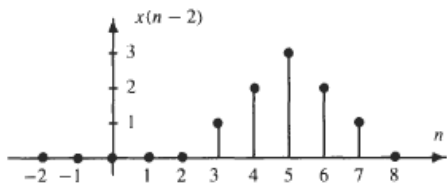
$$y(n) = \begin{cases} x\left(\frac{n}{N}\right) & n = 0, \pm N, \pm 2N, \dots \\ 0 & \text{otherwise} \end{cases}$$

(this operation is known as up-sampling).

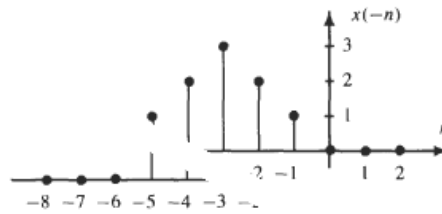
Examples of shifting, reversing, and time scaling a signal are illustrated in Fig. below.



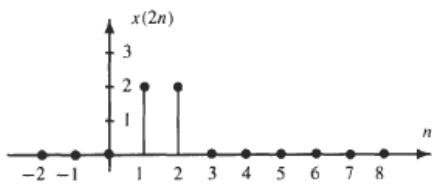
(a) A discrete-time signal.



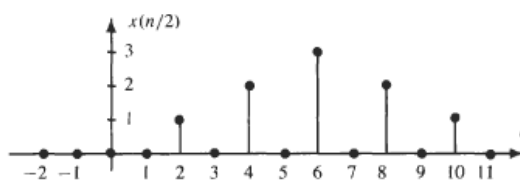
(b) A delay by $n_0 = 2$.



(c) Time reversal.

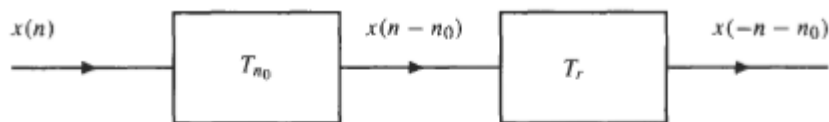


(d) Down-sampling by a factor of 2.

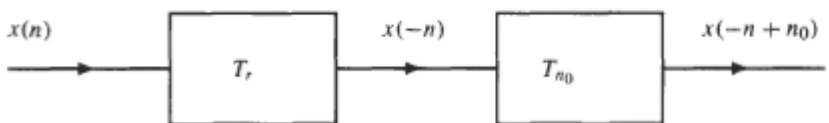


(e) Up-sampling by a factor of 2.

Illustration of the operations of shifting, reversal, and scaling of the independent variable n . Shifting, reversal, and time-scaling operations are order-dependent. Therefore, one needs to be careful in evaluating compositions of these operations. For example, Fig. below shows two systems, one that consists of a delay followed by a reversal and one that is a reversal followed by a delay. As indicated, the outputs of these two systems are not the same.



(a) A delay T_{n_0} followed by a time-reversal T_r .



(b) A time-reversal T_r followed by a delay T_{n_0} .

Example illustrating that the operations of delay and reversal do not commute.

Addition, Multiplication, and Scaling

The most common types of amplitude transformations are addition, multiplication, and scaling. Performing these operations is straightforward and involves only pointwise operations on the signal.

Addition: The sum of two signals

$$y(n) = x_1(n) + x_2(n) \quad -\infty < n < \infty$$

is formed by the pointwise addition of the signal values.

Multiplication The multiplication of two signals

$$y(n) = x_1(n)x_2(n) \quad -\infty < n < \infty$$

is formed by the pointwise product of the signal values.

Scaling Amplitude scaling of a signal $x(n)$ by a constant c is accomplished by multiplying every signal value by c :

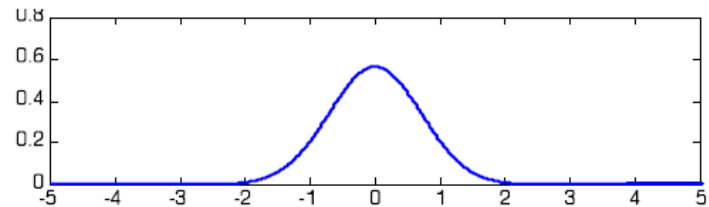
$$y(n) = cx(n) \quad -\infty < n < \infty$$

This operation may also be considered to be the product of two signals, $x(n)$ and $f(n) = c$.

1.7. Temel Sinyaller

1-D:

$$g(x) = e^{-\frac{x^2}{2\sigma^2}}$$

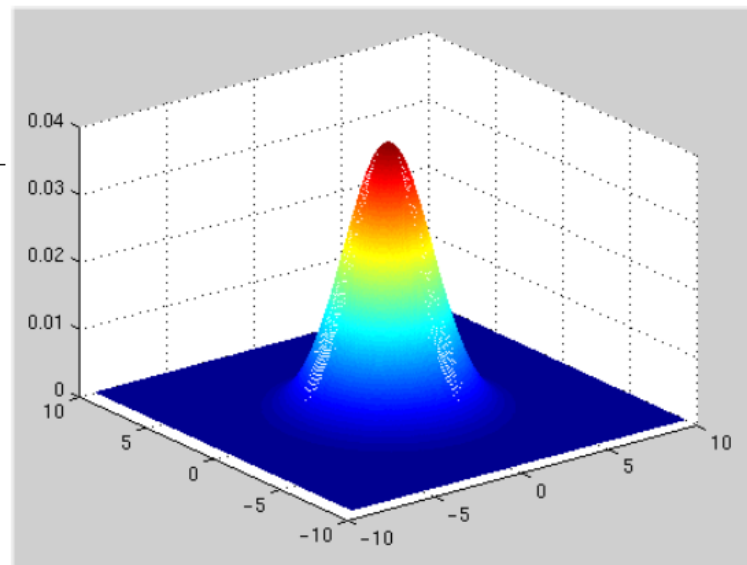


2-D:

$$G(x, y) = e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Slight abuse of notations:
We ignore the normalization
constant such that

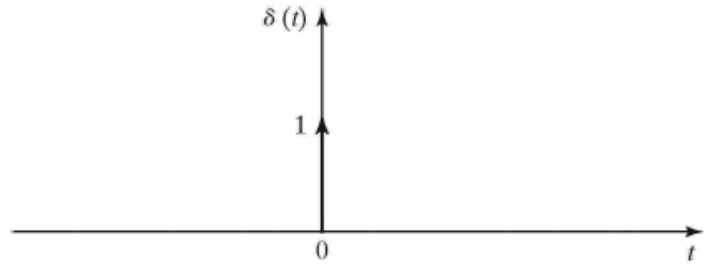
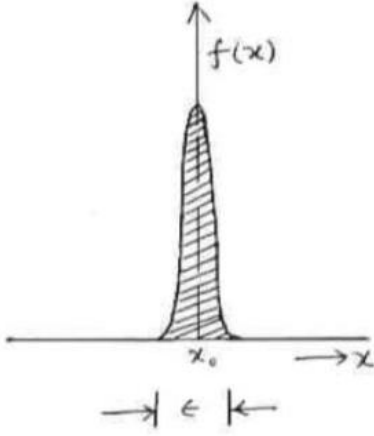
$$\int g(x) dx = 1$$



Dürtü veya Delta İşareti

Matematiksel anlamda, dürtü işareti bir fonksiyon (veya işaret) değildir.

$$\int_{-\infty}^{+\infty} \phi(t)\delta(t)dt = \phi(0)$$



Dürtü İşareti

Dirac Delta Fonksiyonu

$$\delta(-x) = \delta(x)$$

$$x\delta(x) = 0,$$

$$\delta(ax) = a^{-1}\delta(x) \quad (a > 0),$$

$$\delta(x^2 - a^2) = \frac{1}{2}a^{-1}\{\delta(x-a) + \delta(x+a)\} \quad (a > 0),$$

$$\int \delta(a-x) dx \delta(x-b) = \delta(a-b),$$

$$f(x)\delta(x-a) = f(a)\delta(x-a).$$

$$\delta(x,y) = 0 \quad x \neq 0 \text{ \& } y \neq 0$$

$$\iint \delta(x,y) dx dy = 1$$

$$\delta(x,y) = \delta(x)\delta(y).$$

$$\iint \delta(x-a,y-b) f(x,y) dx dy = f(a,b)$$

Delta Dirac (Dürtü) işaretinin temel özellikleri

1. Tüm $t \neq 0$ değerleri için $\delta(t) = 0$ ve $\delta(0) = \infty$

2.

$$x(t) * \delta(t) = x(t)$$

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

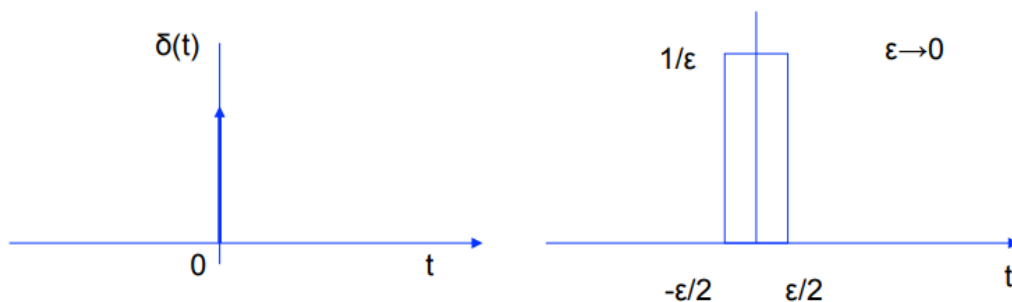
3.

$$\int_{-\infty}^{+\infty} \phi(t) \delta(t - t_0) dt = \phi(t_0)$$

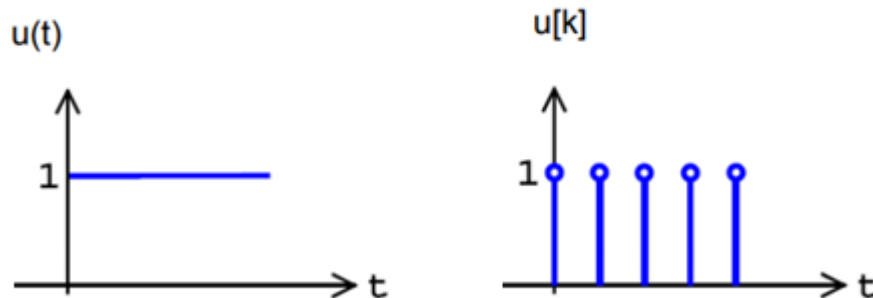
- Unit impulse function

$$\delta(t) = 0 \quad t \neq 0$$

$$\int_{-\infty}^{+\infty} \delta(t) dt = 1$$

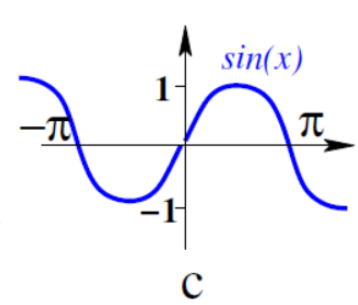
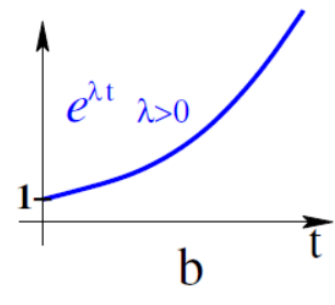
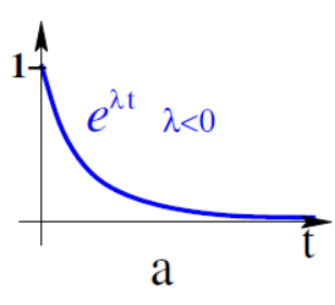
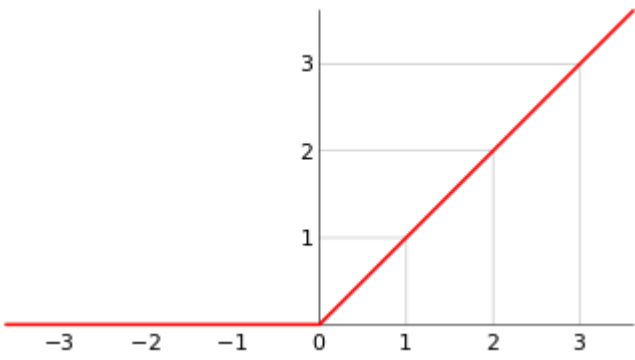
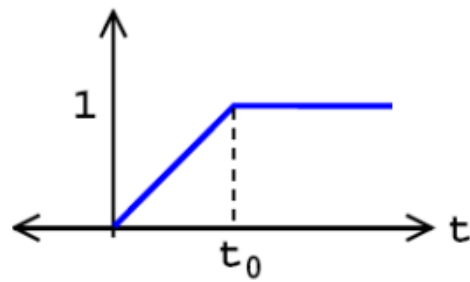


- Continuous and discrete time unit step functions



- Ramp function (continuous time)

$$r(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t}{t_0} & \text{if } 0 \leq t \leq t_0 \\ 1 & \text{if } t > t_0 \end{cases}$$



- Continuous time complex exponential

$$f(t) = Ae^{j\omega t}$$

- Euler's relations

$$Ae^{j\omega t} = A\cos(\omega t) + j(A\sin(\omega t))$$

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- Discrete time complex exponential

– $k=nT$

$$\begin{aligned} f[n] &= Be^{snT} \\ &= Be^{j\omega nT} \end{aligned}$$

- Exponential function e^{st}

– Generalization of the function $e^{j\omega t}$

$$s = \sigma + j\omega$$

Therefore

$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t} = e^{\sigma t}(\cos \omega t + j \sin \omega t)$$

If $s^* = \sigma - j\omega$ (the conjugate of s), then

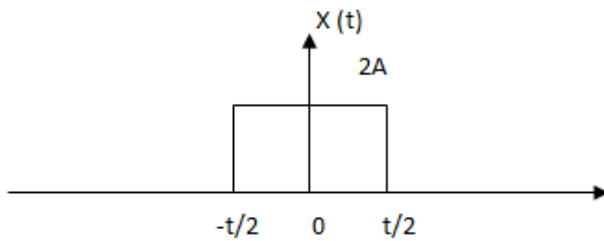
$$e^{s^*t} = e^{\sigma-j\omega t} = e^{\sigma t}e^{-j\omega t} = e^{\sigma t}(\cos \omega t - j \sin \omega t)$$

and

$$e^{\sigma t} \cos \omega t = \frac{1}{2}(e^{st} + e^{s^*t})$$

Soru:

Verilen dikdörtgen sinyalinin matematiksel gösterimi hangisidir?



Explanation: The given rectangular pulse is of amplitude $2A$ for the time interval $-t/2$ to $t/2$ and zero otherwise.

8. $Y(t) = x(2t)$ is _____

- a) Compressed signal
- b) Expanded signal
- c) Shifted signal
- d) Amplitude scaled signal by a factor of 2

Explanation: By comparing the given equation with $y(t) = x(at)$ we get $a=2$. If $a>1$ then it is compressed version of $x(t)$. Answer: a

9. $Y(t) = x(t/5)$ is _____

- a) Compressed signal
- b) Expanded signal
- c) Time shifted signal
- d) Amplitude scaled signal by factor $1/5$

Explanation: $y(t) = x(at)$, comparing this with the given expression we get $a = 1/5$. If $0<a<1$ then it is expanded (stretched) version of $x(t)$. Answer: b

1. The general form of real exponential signal is _____

- a) $X(t) = be^{at}$
- b) $X(t) = (b+1)e^{at}$
- c) $X(t) = b(at)$
- d) $X(t) = be^{(a+1)t}$

Explanation: $X(t) = be^{at}$ is the most general way of representing the exponential signals where both b and a are real parameters. Answer: a

2. In the equation $x(t) = be^{at}$ if $a < 0$, then it is called _____

- a) Growing exponential
- b) Decaying exponential
- c) Complex exponential
- d) Both Growing and Decaying exponential

View Answer

1.8. Temel Sinyallerin Matematiksel Modellenmesi

The elementary functions (of x) include:

- Constant functions: $2, e, \pi$
- Powers of x , x^2, x^3
- Roots of x, \sqrt{x}
- Exponential functions: e^x
- Logarithms: $\log(x)$
- Trigonometric functions: $\sin(x), \cos(x), \tan(x)$
- Inverse trigonometric functions: $\arcsin(x), \arccos(x), \arctan(x)$
- Hyperbolic functions: $\sinh(x), \cosh(x), \tanh(x)$
- Inverse hyperbolic functions: $\operatorname{arcsinh}(x), \operatorname{arccosh}(x), \operatorname{arc tanh}(x)$

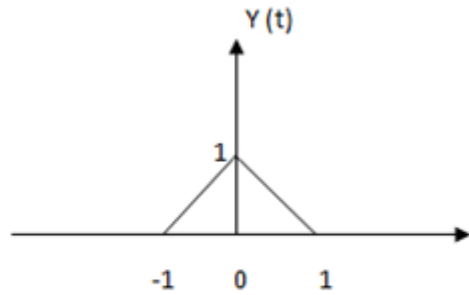
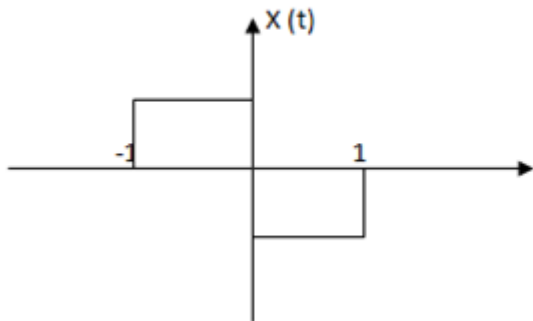
All functions obtained by adding, subtracting, multiplying or dividing any of the previous functions. All functions obtained by composing previously listed functions. Some elementary functions, such as roots, logarithms, or inverse trigonometric functions, are not entire functions and may be multivalued.

Kıyaslama yapılırken aşağıdaki işlemler göz önüne alınır:

- Aritmetik İşlemler: $+, -, *, /$
- İntegral, Türev, Limit
- Dönüşüm: Konvolisyon, Fourier, Laplace, Z

Soru:

The given pair $x(t)$ and $y(t)$ is related by _____



- a) $Y(t) = d/dt(x(t))$
- b) $Y(t) = x(t) + 1$
- c) $Y(t) = \int x(t) .dt$
- d) Not related

View Answer

Answer: c

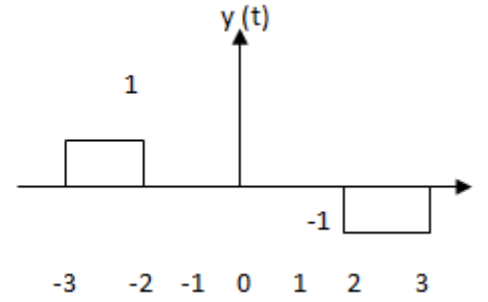
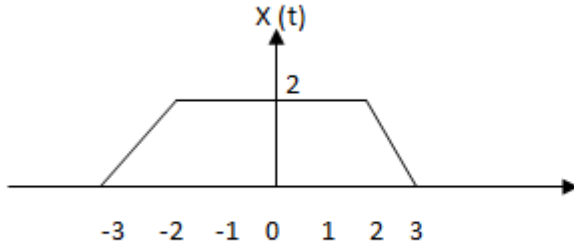
Explanation: The given pair $x(t)$ and $y(t)$ is related by $Y(t) = \int x(t) .dt$. The integral of $x(t)$ gives the $Y(t)$. $Y(t) = 0$ for $t > 1$.

Soru:

Aşağıda verilen $x(t)$ ve $y(t)$ sinyallerinin birbirleri ile ilişkisi nedir?

($F1(t)=at+b$ ve $F2(t)=c$ ifadelerini göz önüne alınız)

- Her iki sinyalin matematiksel ifadesini bulunuz.
- İki sinyalin birbiri ile ilişkisi ne olabilir. (Türev, integral, toplama, çarpma, bölme, çıkarma...)



$$Y(t) = d/dt(x(t))$$

Explanation: The given pair $x(t)$ and $y(t)$ is related by $y(t) = d/dt(x(t))$. From -2 to 2 we have $Y(t)$ is zero because differentiation of constant is zero.

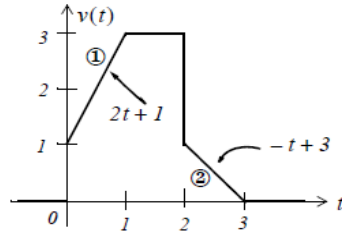


Figure 1.15. Equations for the linear segments of Figure 1.14

Following the same procedure as in the previous examples, we get

$$v(t) = (2t + 1)[u_0(t) - u_0(t-1)] + 3[u_0(t-1) - u_0(t-2)] \\ + (-t + 3)[u_0(t-2) - u_0(t-3)]$$

Multiplying the values in parentheses by the values in the brackets, we get

$$v(t) = (2t + 1)u_0(t) - (2t + 1)u_0(t-1) + 3u_0(t-1) \\ - 3u_0(t-2) + (-t + 3)u_0(t-2) - (-t + 3)u_0(t-3)$$

or

$$v(t) = (2t + 1)u_0(t) + [-(2t + 1) + 3]u_0(t-1) \\ + [-3 + (-t + 3)]u_0(t-2) - (-t + 3)u_0(t-3)$$

and combining terms inside the brackets, we get

$$v(t) = (2t + 1)u_0(t) - 2(t-1)u_0(t-1) - tu_0(t-2) + (t-3)u_0(t-3)$$

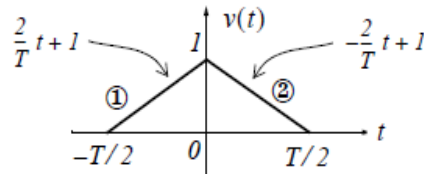


Figure 1.13. Equations for the linear segments of Figure 1.12

For line segment ①,

$$v_1(t) = \left(\frac{2}{T}t + 1\right)\left[u_0\left(t + \frac{T}{2}\right) - u_0(t)\right]$$

and for line segment ②,

$$v_2(t) = \left(-\frac{2}{T}t + 1\right)\left[u_0(t) - u_0\left(t - \frac{T}{2}\right)\right]$$

Combining (1.15) and (1.16), we get

$$v(t) = v_1(t) + v_2(t) \\ = \left(\frac{2}{T}t + 1\right)\left[u_0\left(t + \frac{T}{2}\right) - u_0(t)\right] + \left(-\frac{2}{T}t + 1\right)\left[u_0(t) - u_0\left(t - \frac{T}{2}\right)\right]$$

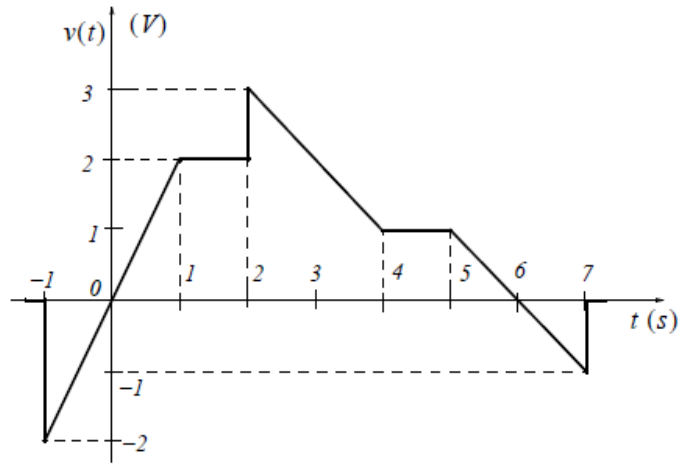


Figure 1.21. Waveform for Example 1.9

Solution:

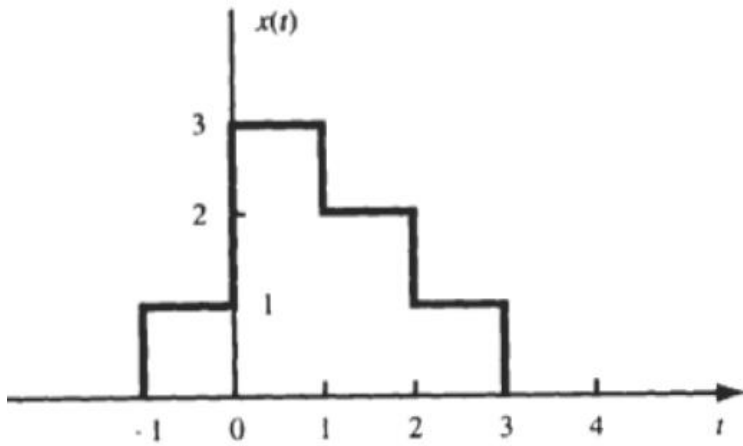
- a. We first derive the equations for the linear segments of the given waveform. These are shown in Figure 1.22.

Next, we express $v(t)$ in terms of the unit step function $u_0(t)$, and we get

$$\begin{aligned}
 v(t) = & 2t[u_0(t+1) - u_0(t-1)] + 2[u_0(t-1) - u_0(t-2)] \\
 & + (-t+5)[u_0(t-2) - u_0(t-4)] + [u_0(t-4) - u_0(t-5)] \\
 & + (-t+6)[u_0(t-5) - u_0(t-7)]
 \end{aligned} \tag{1.52}$$

Soru:

Şekil'de gösterilen sinyalleri birim adım fonksiyonları açısından ifade edin.



$$x(t) = u(t+1) + 2u(t) - u(t-1) - u(t-2) - u(t-3)$$

2. Sistemler

Bir sistemin özellikleri o sistemin davranışları açısından bize önemli bilgiler verir. Böylece sistemleri daha iyi anlayabilir ve analiz edebiliriz. Bu özellikler,

- 1) Deterministik sistemler
- 2) Statik ve Dinamik Sistemler
- 3) Stokastik sistemler
- 4) Sürekli-zaman ve ayrık-zaman sistemler
- 5) Analog ve dijital sistemler
- 6) Invertible/terslenebilir sistemler
- 7) Lineer ve Lineer olmayan sistemler
- 8) Bellekli (dinamik) ve belleksiz (anlık, instantaneous) sistemler
- 9) Causal (nedensel) ve causal olmayan (noncausal) sistemler
- 10) Zamandan bağımsız ve bağımlı sistemler
- 11) Kontrol Sistemleri: kontrol edilebilir – Gözlenebilir Sistemler
- 12) Stable and Unstable Systems

2.1. Bellekli ve Belleksiz Sistemler

Bir sistemin çıkışı, girişin sadece o andaki değerine bağlı ise bu sisteme belleksiz sistem denir. Bir sistemin çıkışı, girişin önceki ve/veya sonraki değerlerine bağlı ise bu sisteme bellekli sistem denir. Gecikme, öteleme sinyalin girişi ile ilgili olmadığı durumlar için sistem belleksizdir. Çünkü zamanda gecikme ya da öteleme sinyalin girişi değildir.

Örnek: $y(t) = ax^2(t) + bx(t)$

Yukarda verilen Sistem çıkış sağlamak için girişin sadece o andaki değerine ihtiyaç duyar, o değer karesini alır, gerekli işlemleri yaparak bize sonucu sunar. Sistemin çıkışı, girişin sadece o andaki değerlerine bağlı olduğundan sistem belleksizdir.

Örnek:

$$u[n] = \sum_{m=-\infty}^n \delta[m]$$

Bu sistemde çıkışın “n” adımıdaki değerini bulabilmek için girişin o ana kadarki bütün değerlerine ihtiyaç vardır. Bu durumda sistemin bir bellek ihtiyacı duyduğu açıktır.

Örnek:

$$y(t) = x^2(t + 1)$$

Bu sistemin çıkış sinyali $y(t)$, giriş sinyali $x(t)$ 'nin sonraki değerlerine bağlı olarak değiştiği için, bu sistem bellekli sistemdir.

Örnek: $y(t + 1) = x^2(t + 1)$

Bu sistemin çıkış sinyali $y(t)$, giriş sinyali $x(t)$ 'nin sadece o anki değerine bağlı olarak değişmektedir. Dolayısıyla sistem belleksiz bir sistemdir.

Örnek: $y(t) = x(t) + \cos(t + \tau)$

Sistemin çıkışı, sistemin girişinin sadece o anki değerine bağlı olarak değiştiği için, söz konusu sistem belleksiz sistemdir. Burada kosinüs terimindeki τ kadar gecikme, giriş sinyali ile ilgili olmadığı için sistemin bellek kontrolünde dikkate alınmamaktadır. Zamanda geciken giriş sinyali değildir.

Bir sistemin çıkışı, girişin sadece o andaki değerine bağlı ise bu sisteme belleksiz sistem denir. Bir sistemin çıkışı, girişin önceki ve/veya sonraki değerlerine bağlı ise bu sisteme bellekli sistem denir. Zamanın fonksiyonu olmayan değişkenler ve sabitler giriş sinyali değildir.

- $y(t) = ax^2(t) + bx(t)$, Belleksiz
- $u[n] = \sum_{n=-\infty}^n \delta[n]$, Belleksiz
- $y(t) = x^2(t + 1)$, bellekli
- $y(t + 1) = x^2(t + 1)$, belleksiz
- $y(t) = x(t) + \cos(t + \emptyset)$, belleksiz, çünkü gecikme, giriş sinyali ile ilgili olmadığı için sistem belleksizdir.

Belleksiz sistemlere örnekler:

“t” nin herhangi bir anında çıkış değeri, “t” nin aynı andaki giriş değerine bağlı ise sistem belleksizdir. Bir belleksiz sistem her zaman nedenseldir, tersi, elbette, her zaman doğru değildir.

$$y(t) = \sin(x(t)), \quad y(t) = \int_{-\infty}^t x(\tau) d\tau, \quad y(t-2) = 3x(t-2)$$

Sistem belleği ile ilgili olarak, sistemler anlık veya dinamik olarak sınıflandırılır. Bir sistem, bir anlık anında çıkışı sadece o andaki girişin bir fonksiyonuysa, anlaktır(hafıza yok).

Fark denklemi ile karakterize edilen sistem $y(n) = 2x(n)$, hafızasız bir sistemdir. Bir örnek, yalnızca dirençlerden oluşan bir elektrik devresidir.

İndüktörler ve kapasitörler gibi depolama elemanlarına sahip herhangi bir sistem dinamik bir sistemdir, çünkü böyle bir sistemin bir anlık çıkışı, girişin geçmiş değerlerinin bir fonksiyonudur.

Bu tür sistemlerin ayrık modelinde, $x(n-1)$ veya $x(n-2)$ gibi, uygulanacak bellek birimlerini gerektiren terimler olacaktır. Çıkış, yalnızca sınırlı sayıda geçmiş giriş örneğine bağlıysa, buna **sonlu bellek sistemi** denir.

Örneğin, $y(n) = x(n-1) + x(n-2)$ bir sistemin iki bellek birimine sahip olan fark denklemidir. Kapasitif veya endüktif elementli sistemler **sonsuz bellek sistemleridir**, çünkü çıktıları girişin tüm geçmişinin bir işlevidir. Anlık sistemler, sıfır belleğe sahip özel bir dinamik sistem durumudur.

2.2. Nedensellik (Causality)

Sistemin herhangi bir zamandaki çıkışı,

- Mevcut zamandaki girdi değerlerine veya
- Geçmişte girdi değerlerine veya
- Mevcut zamanla birlikte geçmişteki girdi değerlerine bağlıysa, bu sistem nedenseldir.

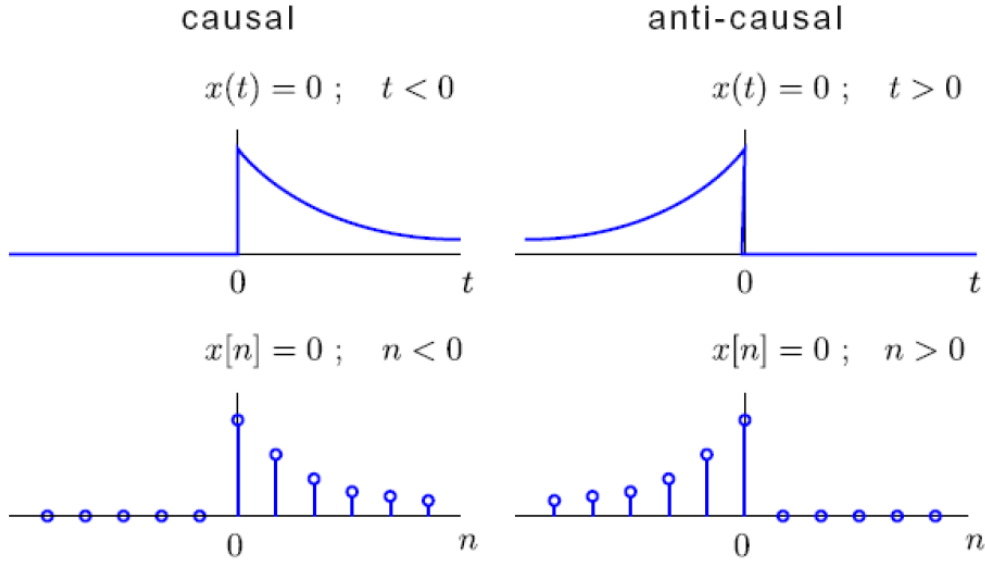
Diğer bir anlatımla eğer bir sistemin çıkışı, şu anki girişi dâhil olmak üzere önceki değerlerine bağlı ise sisteme nedensel (causal) sistem denir. Tüm gerçek zamanlı fiziksel sistemler nedenseldir; çünkü zaman sadece ileriye akar.

$y(t) = x(t-1)$ Nedensel sistem (ve bellekli sistem)

$y(t) = x(t)$ Nedensel sistem

$y(t) = x(t+1)$ Nedensel olmayan sistem





Nedensel sinyallere örnek olarak şunları verebiliriz:

Sistemin herhangi bir zamandaki çıktısı, sadece mevcut zamanda veya geçmişte girdi değerlerine bağlıysa, bu sistem nedenseldir. Diğer bir anlatımla eğer bir sistemin çıkışı, şu anki girişi dâhil olmak üzere önceki değerlerine bağlı ise sisteme nedensel (causal) sistem denir. Tüm gerçek zamanlı fiziksel sistemler nedenseldir; çünkü zaman sadece ileriye akar. Tüm belleksiz sistemler nedenseldir. Kayıtlı veriler üzerinde işlem yapmamız durumunda, nedensellik de gerekli olmayabilir.

- a) $y(t)=x(t-3)-x(t)/3$, nedensel
- b) $y(n) = n x(n)$, nedensel
- c) $y(t) = e^{x(t)}$, nedensel
- d) $y(t) = x(t+1)$, nedensel değil

Örnek:

$$y(t)=x(t-3)-x(t)/3$$

$y(t)$, $x(t)$ 'nin şu anki ve "t-3" anındaki değerine bağlı olduğu için nedenseldir.

Örnek:

$$y(n) = n x(n)$$

$y(n)$, şu anki n ve $x(n)$ değerine bağlıdır. Nedensel.

Örnek:

$$y(t) = e^{x(t)}$$

, nedensel sistemdir.

Örnek:

$$y(t) = x(t+1)$$

Nedensel olmayan sistemdir.

Örnek:

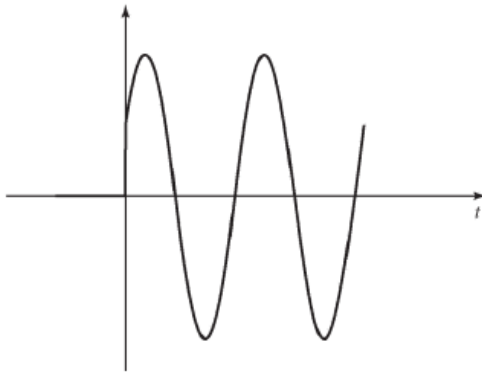
$y(n) = x(n) + n x(n + 1)$, şu anki ve gelecekteki değerlerine bağlıdır; nedensel değil.

Örnek:

$h(t) = e^{-t}u(t)$, şu anki değerlere bağlı nedenseldir.

Örnek:

$$x(t) = \begin{cases} A \cos(2\pi f_0 t + \phi) & t \geq 0 \\ 0 & t < 0 \end{cases}$$



Nedensel bir işaret örneği

Örnek: $y(n) - 1/4y(n-1) - 3/8y(n-2) = -x(n) + 2x(n-1)$, sistem nedenseldir.

Örnek: $y(t) = x(t-2) + x(2-t)$, sistem nedenseldir.

Örnek: $y(t) = dx(t)/dt$, Sistem nedenseldir.

Örnek: $y(t) = x(t/3)$, sistem nedensel değildir. Çünkü, $t=-3$ alındığında, $y(-3)=x(-1)$; -1 değeri -3 değerine göre gelecektir.

Örnek: $y(t)=\cos(x(t))$, nedenseldir.

Nedensel sistemlere örnekler:

$$y(t) = 3x(t-2), \quad y(t) = \int_{-\infty}^t x(\tau) d\tau, \quad y(t) = x^3(t)$$

Nedensel olmayan sistemlere örnekler:

$$y(t) = x(2), \quad y(t) = 3x(t+2), \quad y(t) = \int_{-\infty}^{t+1} x(\tau) d\tau$$

2.3. Kararlılık (Stability)

Sınırlı girişe Sınırlı Çıkış (BIBO) özelliğine sahip olabilmesi için $|x(t)| \leq Bx < \infty$ girişi uygulandığında çıkış, $|y(t)| \leq Bx < \infty$ olmalıdır.

Stabilite için birkaç tanım var. Burada sınırlı giriş bağlantılı çıkış (Bounded input bonded output - BIBO) kararlılığını dikkate alacağız. Her sınırlanmış giriş sınırlı bir çıkış üretiyorsa, bir sistem BIBO kararlı olarak söylenir.

Eğer tüm "n" değerleri için $|x[n]| < M < \infty$ koşulu sağlanıyorsa $x(n)$ sınırlıdır.

Örnek:

Aşağıdaki eşitlik ile tanımlı ortalama alan bir sistem kararlıdır, Çünkü sınırlı sayıdaki değerlerin toplamı sınırlıdır.

$$y[n] = \frac{1}{2N+1} \sum_{k=-N}^N x[k]$$

Örnek:

The accumulator system defined by

$$y[n] = \sum_{k=-\infty}^n x[k]$$

is unstable. If we take $\{x[n]\} = \{u[n]\}$, the unit step then $y[0] = 1$, $y[1] = 2$, $y[2] = 3$, $y[n] = n + 1$, $n \geq 0$ so $y[n]$ grows without bound.

Örnek:

$$h_1(t) = \exp[-(1 - 2j)t] u(t)$$

- Sistem nedenseldir.
- Sistem kararlıdır, çünkü $t \rightarrow \infty \rightarrow h_1(t) \rightarrow 0$

2.4. Tersine Çevrilebilir

Bir sistem birkaç giriş için aynı çıkışlara sahipse, doğru girişi bulmak imkansızdır. Bu nedenle, bir sistem farklı girdilere farklı çıktılar verirse tersine çevrilebilir. Birçok giriş için aynı çıkış verirse ters çevrilemez. Hangisi ters çevrilemez bir sistemdir? $y[n] = 0$

2.5. Doğrusallık – Linear Sistemler

Doğrusal, Zamanda Değişmez Sistemler - Linear, Time Invariant (LTI) Systems

Doğrusal (linear) bir sistem çok basit iki özelliğe sahiptir: Toplanırlık (additivity) ve Homojenite (homogeneity).

If difference equation has with constant coefficients; therefore, it is linear and time-invariant.

- Toplanırlık:

Sisteme $x_1(t)$ sinyali girildiğinde $y_1(t)$ sinyali ve $x_2(t)$ sinyali girildiğinde $y_2(t)$ sinyali elde ediliyorken; $ax_1(t)+bx_2(t)$ sinyali girildiğinde $ay_1(t)+by_2(t)$ sinyali elde ediyor ise sistem doğrusallığın ilk basamağını geçmiştir.

b) Homojenite:

Sisteme $x_1(t)$ sinyali girildiğinde $y_1(t)$ sinyalini elde ediliyorken, $ax_1(t)$ sinyali girildiğinde $ay_1(t)$ sinyalini elde ediyor ise (a herhangi bir sabit olmak üzere) sistem artık doğrusaldır diyebiliriz.

Bu durumda; a ve b şıkları birleştirilerek, süperpozisyon (superposition) özelliğini sağlayan sistemler doğrusal sistemlerdir. Diğer bir deyişle; $ax_1(t) + bx_2(t)$ sinyali girildiğinde $ay_1(t) + by_2(t)$, (a ve b herhangi iki sabit olmak üzere) çıkış sinyalinin elde edildiği sistemlere doğrusal sistemler denir.

Soru:Bir sistemin lineer özelliğini yazınız. $F(x)=3x^2+2$ sistemi lineer mi?

Doğrusal (linear) bir sistem çok basit iki özelliğe sahiptir: Toplanırlık (additivity) ve Homojenite (homogeneity).

Sisteme $x_1(t)$ sinyali girildiğinde $y_1(t)$ sinyali ve $x_2(t)$ sinyali girildiğinde $y_2(t)$ sinyali elde ediliyorken; $ax_1(t)+bx_2(t)$ sinyali girildiğinde $ay_1(t)+by_2(t)$ sinyali elde ediyor ise sistem doğrusaldır.

$$F1(ax)=3a^2 x^2+2$$

$$F2(bx)=3b^2 x^2+2$$

$$F(ax+bx)=3(ax+bx)^2+2$$

$$F1(ax)+ F2(bx)= 3a^2 x^2+3b^2 x^2 + 4$$

$F(ax+bx)$ ifadesi $F1(ax)+ F2(bx)$ ifadesine eşit olmadığından lineer değildir.

Örnek: $y(t) = x(t) + \cos(t + \tau)$

Bu sisteme $x(t)=ax_1(t)+bx_2(t)$ girişi uyguladığımızda, Çıkış,

$$y(t)=x(t)+\cos(t+\tau)=(ax_1(t)+bx_2(t))+\cos(t+\tau)$$

olarak elde edilir. Bu durumda sistemin doğrusal olduğunu söyleyemez.

Next we consider linearity. Roughly speaking, a system is linear if its behavior is scale-independent; a result of this is the superposition principle. More precisely, suppose that $y_1(t) = F[u_1(t)]$ and $y_2(t) = F[u_2(t)]$. Then linearity means that for any two constants α_1 and α_2 ,

$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t) = F[\alpha_1 u_1(t) + \alpha_2 u_2(t)].$$

A simple special case is seen by setting $\alpha_2 = 0$:

$$y(t) = \alpha_1 y_1(t) = F[\alpha_1 u_1(t)],$$

making clear the scale-invariance. If the input is scaled by α_1 , then so is the output. Here are some examples of linear and nonlinear systems:

$$\begin{aligned} y(t) &= c \frac{du}{dt} \text{ (linear and time-invariant)} \\ y(t) &= \int_0^t u(t_1) dt_1 \text{ (linear but not time-invariant)} \\ y(t) &= 2u^2(t) \text{ (nonlinear but time-invariant)} \\ y(t) &= 6u(t) \text{ (linear and time-invariant).} \end{aligned}$$

Input output relationship

$$y(t) = \int_{-\infty}^{5t} x(\tau) d\tau, \quad t > 0$$

Causality :

$y(t)$ depends on $x(5t)$, $t > 0$ system is non-causal.

For example $t = 2$

$y(2)$ depends on $x(10)$ (future value of input)

Linearity :

Output is integration of input which is a linear function, so system is linear.

Soru: Aşağıdaki sistemlerin her biri için, bunun olup olmadığını belirleyin:

(i) kararlı

(ii) zaman değişmez

(iii) doğrusal

(iv) tersine çevrilebilir

$$y(t) = x(2t + 1) + 1, \text{ for input } x(t) \text{ and output } y(t).$$

- i. Stable. If $|x(t)| \leq M \forall t$, then $|y(t)| \leq M + 1 \forall t$.
- ii. Not time-invariant. Let $x'(t) = x(t - t_0)$. Then $y'(t) = x'(2t + 1) + 1 = x(2t + 1 - t_0) + 1$. However $y(t - t_0) = x(2t - 2t_0 + 1) + 1$.
- iii. Not linear. Let $y_1(t) = x_1(2t + 1) + 1$, and $y_2(t) = x_2(2t + 1) + 1$.
Let $x(t) = ax_1(t) + bx_2(t)$.
Then $y(t) = x(2t + 1) + 1 = ax_1(2t + 1) + bx_2(2t + 1) + 1$.
However $ay_1(t) + by_2(t) = ax_1(2t + 1) + bx_2(2t + 1) + a + b$
- iv. Invertible. Observe that $x(t) = y(\frac{t-1}{2}) - 1$.

Soru:

Aşağıdaki sistem: i) doğrusal, ii) zamanla değişmeyen, iii) BIBO kararlı, iv) tersine çevrilebilir, v) hafıza kaybı, vi) nedensel mi? Her biri için kısa bir gerekçe üretin.

$$y(t) = x^2(t)$$

Linear: no, since it doesn't satisfy scaling. $\alpha x(t)$ input produces $\alpha^2 x(t)$ output.

Time-invariant: yes, since $y(t - t_0) = x(t - t_0)^2$

BIBO stable: yes, since if $|x(t)| \leq B_1 < \infty$ then $|y(t)| \leq B_1^2 \leq \infty$.

Invertible: no, since any negative input value is lost, and we can not recover the original sign of the input.

Memoryless: yes, since $y(t)$ depends on no other value of $x(t)$ other than at time t .

Causal: yes, since all memoryless systems are causal.

Soru:

Aşağıdaki sistem: i) doğrusal, ii) zamanla değişmeyen, iii) BIBO kararlı, iv) tersine çevrilebilir, v) hafıza kaybı, vi) nedensel mi? Her biri için kısa bir gerekçe üretin.

$$y(t) = t^2 x(t)$$

Linear: yes, since with input $a_1 x_1(t) + a_2 x_2(t)$ produces output $t^2(a_1 x_1(t) + a_2 x_2(t)) = a_1 t^2 y_1(t) + a_2 t^2 y_2(t) = a_1 y_1(t) + a_2 y_2(t)$ using the usual definitions of $y_1(t)$ and $y_2(t)$.

Time-invariant: no, since the system depends on values of $x(t)$ at specific values of time via the t^2 factor. I.e., shifting the input produces output $t^2 x(t - t_0)$ but shifting the output produces $(t - t_0)^2 x(t - t_0)$.

BIBO stable: no, since the output is unbounded for many bounded inputs (e.g., consider $x(t) = u(t)$, then $y(t) = t^2 u(t)$ grows in t without bound).

Invertible: No, since the value at $t = 0$ is lost. I.e., $y(0) = 0$ regardless of $x(0)$, so we can't get back $x(0)$.

Memoryless: yes, since $y(t)$ depends values of $x(t)$ only at time t .

Causal: yes, all memoryless systems are causal.

Soru:

Aşağıdaki sistem: i) doğrusal, ii) zamanla değişmeyen, iii) BIBO kararlı, iv) tersine çevrilebilir, v) hafıza kaybı, vi) nedensel mi? Her biri için kısa bir gerekçe üretin.

$$y(t) = e^{j\omega_0 t} x(t) - jx(t - t_0)$$

Burada t_0 ve ω_0 sabitlerdir.

Linear: yes, since $S[\alpha_1 x_1(t) + \alpha_2 x_2(t)] = \alpha_1 y_1(t) + \alpha_2 y_2(t)$. The reason this is true is that

$$S[\alpha_1 x_1(t) + \alpha_2 x_2(t)] = e^{j\omega_0 t} (\alpha_1 x_1(t) + \alpha_2 x_2(t)) - j (\alpha_1 x_1(t - t_0) + \alpha_2 x_2(t - t_0)) \quad (0.4)$$

$$= \alpha_1 (e^{j\omega_0 t} x_1(t) - j x_1(t - t_0)) + \alpha_2 (e^{j\omega_0 t} x_2(t) - j x_2(t - t_0)) \quad (0.5)$$

$$= \alpha_1 y_1(t) + \alpha_2 y_2(t) \quad (0.6)$$

Time-invariant: No, since when we shift the output $y(t - t_0)$ we'll shift the $e^{j\omega_0 t}$ factor in the first term, but this won't happen when we shift the input.

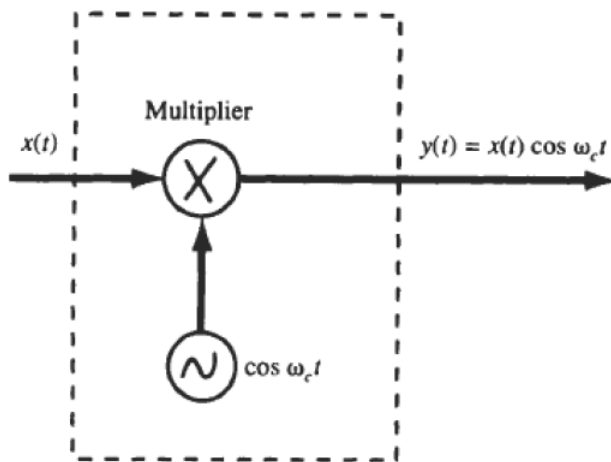
BIBO stable: yes, since nothing is going to make the output infinite as long as $x(t)$ is finite.

Memoryless: No, since $y(t)$ depends on values of the input at times other than time t .

Causal: yes, since $y(t)$ depends on values of $x(t)$ only up to time t .

Soru:

Şekilde gösterilen sistemin (a) hafızaya sahip mi, (b) nedensel mi, (c) doğrusal mı, (d) zamanla değişmeyen mi veya (e) kararlı olup olmadığını belirleyin.



$y(t) = T\{x(t)\} = x(t) \cos \omega_c(t)$; Since the value of the output $y(t)$ depends on only the present values of the input $x(t)$, the system is **memoryless**.

E.g. $t = 5$, $y(5) = x(5) \cos \omega_c(5)$; Since the output $y(t)$ does not depend on the future values of the input $x(t)$, the system is **causal**.

(c) Let $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$. Then

$$\begin{aligned}y(t) &= [\alpha_1 x_1(t) + \alpha_2 x_2(t)] \cos \omega_c t \\&= \alpha_1 x_1(t) \cos \omega_c t + \alpha_2 x_2(t) \cos \omega_c t \\&= \alpha_1 y_1(t) + \alpha_2 y_2(t)\end{aligned}$$

Thus, the system is **linear**.

(d) Let $y_1(t)$ be the output produced by the shifted input $x_1(t) = x(t - t_0)$. Then

$$y_1(t) = T\{x(t - t_0)\} = x(t - t_0) \cos \omega_c(t)$$

But $y(t - t_0) = x(t - t_0) \cos \omega_c(t - t_0) \neq y_1(t)$. Hence, the system is **not time-invariant**.

Since $|\cos \omega_c t| \leq 1$, we have

$$|y(t)| = |x(t) \cos \omega_c t| \leq |x(t)|$$

Thus, if the input $x(t)$ is bounded, then the output $y(t)$ is also bounded and the system is **BIBO stable**.

Soru.

a) $\frac{dy}{dt} + 6y(t) = 4x(t)$

This is an ordinary differential equation with constant coefficients, therefore, it is linear and time-invariant. It contains memory and it is causal.

b) $\frac{dy}{dt} + 4ty(t) = 2x(t)$

This is an ordinary differential equation. The coefficients of $4t$ and 2 do not depend on y or x , so the system is linear. However, the coefficient $4t$ is not constant, so it is time-varying. The system is also causal and has memory.

c) $y[n] + 2y[n - 1] = x[n + 1]$

This is a difference equation with constant coefficients; therefore, it is linear and time-invariant. It is noncausal since the output depends on future values of x . Specifically, let $x[n] = u[n]$, then $y[-1] = 1$.

d) $y(t) = \sin(x(t))$

check linearity:

$$y_1(t) = \sin(x_1(t))$$

$$y_2(t) = \sin(x_2(t))$$

Solution to an input of $a_1x_1(t) + a_2x_2(t)$ is $\sin(a_1x_1(t) + a_2x_2(t))$.

This is not equal to $a_1y_1(t) + a_2y_2(t)$.

As a counter example, consider $x_1(t) = \pi$ and $x_2(t) = \pi/2$, $a_1 = a_2 = 1$

the system is causal since the output does not depend on future values of time, and it is memoryless
the system is time-invariant

$$f) \quad y[n + 1] + 4y[n] = 3x[n + 1] - x[n]$$

Rewrite the equation as $y[n] + 4y[n - 1] = 3x[n] - x[n - 1]$ by decreasing the index.

This is a difference equation with constant coefficients, so it is linear and time-invariant. The output does not depend on future values of the input, so it is causal. It has memory.

$$h) \quad y[n] = x[2n]$$

has memory since the output relies on values of the input at other than the current index n ,

causal? Let $x[n] = u[n-2]$, so $x[1] = 0$. Then $y[1] = x[2] = 1$, so not causal.

linear? Let $y_1[n] = x_1[2n]$ and $y_2[n] = x_2[2n]$. The response to an input of $x[n] = ax_1[n] + bx_2[n]$ is

$$y[n] = ax_1[2n] + bx_2[2n], \text{ which is } ay_1[2n] + by_2[2n], \text{ so this is linear}$$

time-invariant: Let $y_1[n]$ represent the response to an input of $x[n-N]$, so $y_1[n] = x[2(n-N)]$. This is also equal to $y[n-N]$, so the system is time-invariant.

$$i) \quad y[n] = nx[2n]$$

This is similar to part h), except for the n coefficient. Similar to above, it is noncausal, has memory and is linear. Check time-invariance:

Let $y_1[n]$ represent the response to an input of $x[n-N]$, so $y_1[n] = nx[2(n-N)]$. This is not equal to $y[n-N] = (n-N)x[2(n-N)]$, so the system is time-varying.

$$j) \quad \frac{dy}{dt} + \sin(t)y(t) = 4x(t)$$

This is an ordinary differential equation with coefficients $\sin(t)$ and 4. Neither depends on y or x , so it is linear. However, the explicit dependence on t means that it is time-varying. It is causal and has memory.

$$k) \quad \frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 4y(t) = \frac{dx}{dt} + 4x(t)$$

This is an ordinary differential equation with constant coefficients, so it is linear and time-invariant. It is also causal and has memory.

System	Linear	Time-invariant	Causal
$y(t) = 3x(t) \cos(t)$	yes	no	yes
$y(t) = \sqrt{x^2(t)}$	no	yes	yes
$y[n] = \begin{cases} +1, & x[n] \geq 0 \\ -1, & x[n] < 0 \end{cases}$	no	yes	yes
$y(t) = \int_t^{t+1} x(\lambda) d\lambda$	yes	yes	no
$y[n] = 2(x[n+1]u[n] - x[n]) + 1$	no	no	no

2.6. Zamandan Bağımsız (Time Invariant)

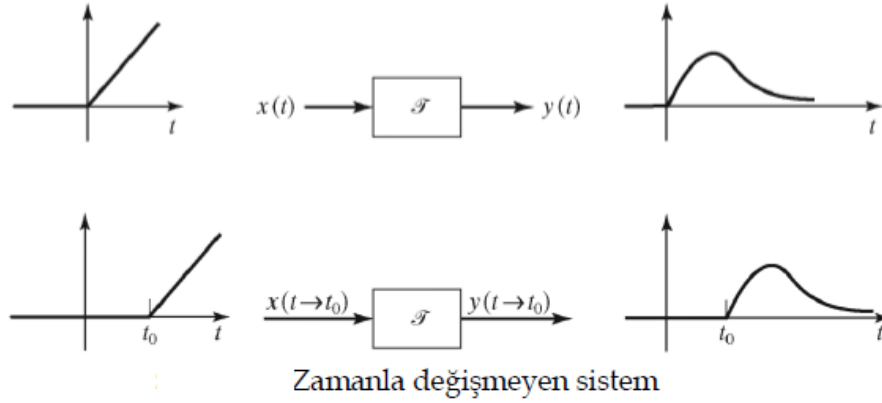
Zamanda değişmezlik ise bir sistemin girişinde t_0 kadar gecikme oluyorsa, çıkışında da t_0 kadar gecikme olmasıdır. Bu durumda sistem zamanda değişmezdir (time invariant). Bir sisteme $x(t)$ sinyalini girdiğimizde $y(t)$ sinyalini elde ediyorken, $x(t - t_0)$ sinyalini girdiğimizde çıkış aynı miktarda gecikerek $y(t - t_0)$ çıkış sinyalini veriyorsa sistem zamanda değişmez (time invariant) bir sistemdir.

Bir sistemin zamandan bağımsız olabilmesi için davranışı zamanın ne olduğuna bağlı olmamalıdır. Yani girdi sinyalinde yaptığımız tüm kaydırma işlemleri çıktı sinyaline yansmalıdır. Örneğin

$$x(t) \rightarrow y(t), x(t - t_0) \rightarrow y(t - t_0)$$

Bir sistem ancak ve ancak, tüm $x(t)$ ve tüm t_0 değerleri için, $x(t-t_0)$ için sistem yanıtı $y(t-t_0)$ ise zamanla değişmeyen bir sistemdir. Burada $y(t)$, sistemin $x(t)$ için üretmiş olduğu yanıtıdır.

Sistemin davranışı ve özellikleri zamanla değişmezse, bir sistemin zamanla değişmez olduğu söylenir. Böylece, bir zaman gecikmesi veya giriş sinyaldeki zaman ilerlemesi, çıkış sinyalinde aynı gecikmeye veya ilerlemeye yol açarsa, bir sistemin zamanla değişmez olduğu söylenir.



Doğrusal Zamanla Değişmeyen sistem kümesi bazı nedenlerden dolayı özellikle önemlidir. Bu sistemlerin girişlerine gösterdikleri yanıt, basit bir şekilde giriş işareti ile sistemin birim dürtü yanıtının evrişimi (konvolüsyonu) olarak elde edilebilir.

Soru: Aşağıdaki sistemin özelliklerini belirleyiniz.

$$y(t) = x(t - 2) + x(2 - t)$$

Nedensellik:

$$t = 0 \text{ anında, } y(0) = x(-2) + x(2)$$

Sistemin şu andaki değeri, geçmişteki ve gelecekteki değerine bağlıdır. Bu yüzden sistem nedensel değildir. Çünkü sistemin herhangi bir zamandaki çıktısı, mevcut zamanda veya geçmişte girdi değerlerine bağlıysa, sistem nedenseldir.

Bellekli ya da belleksiz:

Sistem belleklidir. Çünkü t anındaki çıkış t'den farklı zamanlardaki giriş değerlerine bağlıdır.

Doğrusallık, Lineerlik:

Sistemin girişine $x(t)=ax_1(t)+bx_2(t)$ sinyalini girdiğimizde, çıkışta $y(t)=ay_1(t)+by_2(t)$, a ve b herhangi iki sabit olmak üzere, $y(t)$ sinyalini elde ettiğimiz sistemlere doğrusal sistemler denir.

$$y(t) = x(t - 2) + x(2 - t)$$

$$y_1(t) = x_1(t - 2) + x_1(2 - t)$$

$$y_2(t) = x_2(t - 2) + x_2(2 - t)$$

$$x(t) = \alpha x_1(t) + \beta x_2(t)$$

$$y(t) = (\alpha x_1 + \beta x_2)(t - 2) + (\alpha x_1 + \beta x_2)(2 - t)$$

$$y(t) = \alpha x_1(t - 2) + \beta x_2(t - 2) + \alpha x_1(2 - t) + \beta x_2(2 - t)$$

$$y(t) = \alpha x_1(t - 2) + \alpha x_1(2 - t) + \beta x_2(t - 2) + \beta x_2(2 - t)$$

$$y(t) = \alpha(x_1(t - 2) + x_1(2 - t)) + \beta(x_2(t - 2) + x_2(2 - t))$$

$$y(t) = \alpha y_1(t) + \beta y_2(t)$$

sistem lineerdir.

Sistemin zamanla değişip değişmediği:

Bir sistemin zamandan bağımsız olabilmesi için davranışı zamanın ne olduğuna bağlı olmamalıdır. Yani girdi sinyalinde yaptığımız tüm kaydırma işlemleri çıktı sinyaline yansımalıdır. Örneğin

$$x(t) \rightarrow y(t), x(t - t_0) \rightarrow y(t - t_0)$$

Böylece, bir zaman gecikmesi veya giriş sinyaldeki zaman ilerlemesi, çıkış sinyalinde aynı gecikmeye veya ilerlemeye yol açarsa, bir sistemin zamanla değişmez olduğu söylenir.

Örnek: $y(t) = x(t-2)+x(2-t)$

The system is time invariant. $t \rightarrow t - a$

$$y(t - a) = x((t - a) - 2) + x(2 - (t - a))=y(t-a)$$

Time Invariance: The system is time invariant. To see this, we let $y(t)$ be the output corresponding to the input $x(t)$ and let $x_a(t) = x(t - a)$ ¹. Then the output $y_a(t)$ corresponding to the input signal $x_a(t)$ is

$$y_a(t) = x_a(t - 2) + x_a(2 - t) \quad (9)$$

$$= x((t - a) - 2) + x(2 - (t - a)) \quad (10)$$

$$= y(t - a). \quad (11)$$

(e) $y(t) = \cos(x(t))$ (causal, invertible, linear, memoryless, time invariant)

Causality: The system is memoryless, hence causal.

Invertibility: The system is NOT invertible, e.g. suppose that $x_1(t) = (\pi/2)u(t)$ and $x_2(t) = -(\pi/2)u(t)$. Then $y_1(t) = \cos(x_1(t)) = 0 = \cos(x_2(t)) = y_2(t), \forall t$.

Linearity: The system is NOT linear because if

$$y_1(t) = \cos(x_1(t)), \text{ and} \quad (37)$$

$$y_2(t) = \cos(x_2(t)), \quad (38)$$

and $x(t) = \alpha x_1(t) + \beta x_2(t)$, then the output $y(t)$ corresponding to the input $x(t)$ is

$$y(t) = \cos(\alpha x_1(t) + \beta x_2(t)) \quad (39)$$

$$= \cos(\alpha x_1(t)) \cos(\beta x_2(t)) - \sin(\alpha x_1(t)) \sin(\beta x_2(t)) \quad (40)$$

$$\neq \alpha \cos(x_1(t)) + \beta \cos(x_2(t)) \quad (41)$$

in general. (We used an identity from Chapter B of the textbook to get the second equality above.)

Memorylessness: The system is memoryless because the output at time t depends on input values at only time t .

Time Invariance: The system is time invariant. To see this, we let $y(t)$ be the output corresponding to the input $x(t)$ and let $x_a(t) = x(t - a)$. Then the output $y_a(t)$ corresponding to the input signal $x_a(t)$ is

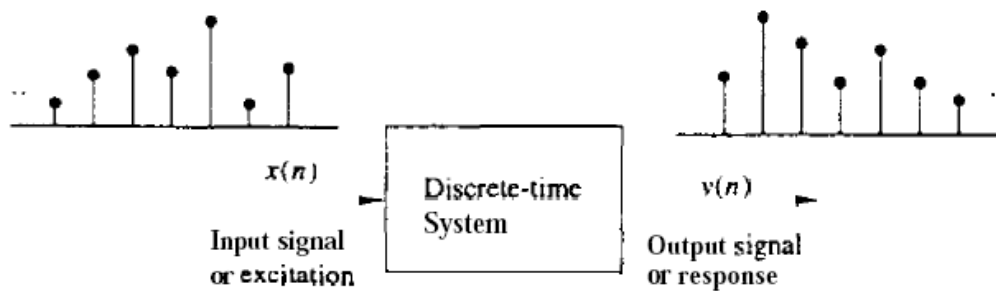
$$y_a(t) = \cos(x_a(t)) = \cos(x(t - a)) = y(t - a). \quad (42)$$

2.7. Ayırık Zamanlı Sistemler

In many applications of digital signal processing we wish to design a device or an algorithm that performs some prescribed operation on a discrete-time signal. Such a device or algorithm is called a discrete-time system. More specifically, a discrete-time system is a device or algorithm that operates on a discrete-time signal, called the input or excitation, according to some well-defined rule, to produce another discrete-time signal called the output or response of the system. In general, we view a system as an operation or a set of operations performed on the input signal $\mathbf{x}(n)$ to produce the output signal $\mathbf{y}(n)$. We say that the input signal $\mathbf{x}(n)$ is transformed by the system into a signal $\mathbf{y}(n)$ and express the general relationship between $\mathbf{x}(n)$ and $\mathbf{y}(n)$

$$\mathbf{y}(n) \equiv \mathcal{T}[\mathbf{x}(n)]$$

where the symbol T denotes the transformation (also called an operator), or processing performed by the system on $x(n)$ to produce $y(n)$. *as shown below*.



There are various ways to describe the characteristics of the system and the operation it performs on $x(n)$ to produce $y(n)$, we shall be concerned with the time-domain characterization of systems. We shall begin with an input-output description of the system. The input-output description focuses on the behavior at the terminals of the system and ignores the detailed internal construction or realization of the system. In this description we develop mathematical equations that not only describe the input-output behavior of the system but specify its internal behavior and structure.

Input-Output Description of Systems

The input-output description of a discrete-time system consists of a mathematical expression or a rule, which explicitly defines the relation between the input and output signals (input-output relationship). The exact internal structure of the system is either unknown or ignored. Thus the only way to interact with the system is by using its input and output terminals (i.e., the system is assumed to be a "black box" to the user). To reflect this philosophy, we use the graphical representation in figure above.

Example.

Determine the response of the following systems to the input signal

$$x(n) = \begin{cases} |n|, & -3 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

- (a) $y(n) = x(n)$
- (b) $y(n) = x(n - 1)$
- (c) $y(n) = x(n + 1)$
- (d) $y(n) = \frac{1}{3}[x(n + 1) + x(n) + x(n - 1)]$
- (e) $y(n) = \max\{x(n + 1), x(n), x(n - 1)\}$
- (f) $y(n) = \sum_{k=-\infty}^n x(k) = x(n) + x(n - 1) + x(n - 2) + \dots$

Solution First, we determine explicitly the sample values of the input signal

$$x(n) = \{ \dots, 0, 3, 2, 1, 0, 1, 2, 3, 0, \dots \}$$

↑

(a) In this case the output is exactly the same as the input signal. Such a system is known as the *identity* system.

(b) This system simply delays the input by one sample. Thus its output is given by

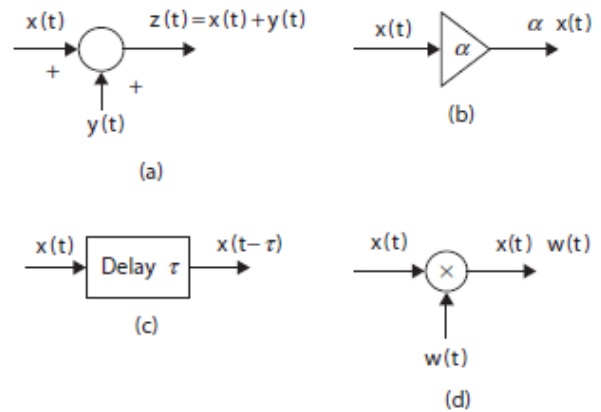
$$x(n) = \{ \dots, 0, 3, 2, 1, 0, 1, 2, 3, 0, \dots \}$$

↑

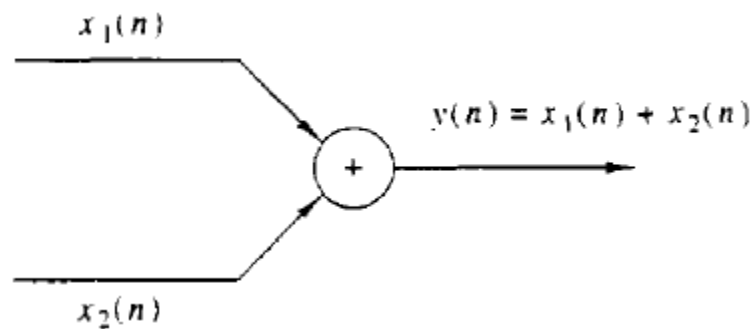
2.8. Sistemlerin Blok Diyagram Gösterimi ve Matematiksel Modellenmesi

It is useful at this point to introduce a block diagram representation of discrete time systems. For this purpose we need to define some basic building blocks that can be interconnected to form complex systems.

Diagrams of basic signal operations: (a) adder, (b) constant multiplier, (c) delay, and (d) time windowing or modulation.



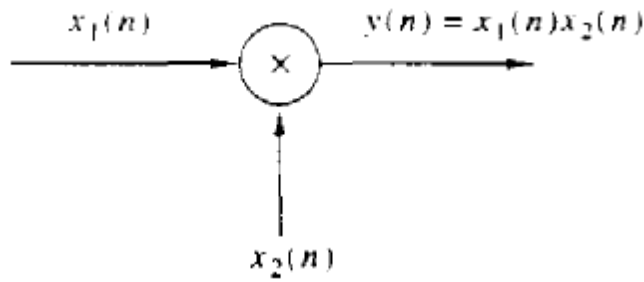
An adder. Figure below illustrates a system (adder) that performs the addition of two signal sequences to form another (the sum) sequence, which we denote as $y(n)$. Note that it is not necessary to store either one of the sequences in order to perform the addition. In other words, the addition operation is *memoryless*.



A constant multiplier. This operation is depicted by Fig. below, and simply represents applying a scale factor on the input $x(n)$. Note that this operation is also memoryless.

$$\underline{x(n) \quad a \quad y(n) = ax(n)}$$

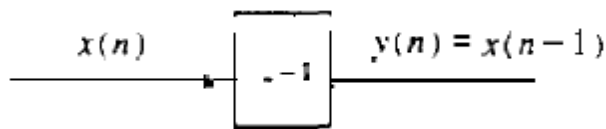
A signal multiplier. Figure below illustrates the multiplication of two signal sequences to form another (the product) sequence, denoted in the figure as $y(n)$. As in the preceding two cases, we can view the multiplication operation as memoryless.



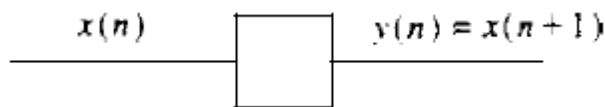
A unit delay element. The unit delay is a special system that simply delays the signal passing through it by one sample. Figure below illustrates such a system. If the input signal is $x(n)$, the output is $x(n - 1)$. In fact, the sample $x(n - 1)$ is stored in memory at time $n - 1$ and it is recalled from memory at time n to form

$$y(n) = x(n - 1)$$

Thus this basic building block requires memory. The use of the symbol z^{-1} to denote the unit of delay will become apparent when we discuss the Z transform



A unit advance element: In contrast to the unit delay, a unit advance moves the input $x(n)$ ahead by one sample in time to yield $x(n + 1)$. Figure below illustrates this operation, with the operator Z being used to denote the unit advance.



We observe that **any** such advance is physically impossible in real time, since. In fact, it involves **looking** into the **future** of the signal. On the other hand. if we store **the signal in the** memory of the computer, we can recall any sample at any time. In such a non real-time application, it is possible to advance the signal $x(n)$ in time.

Example .

The discrete-time system described by the input-output relation.

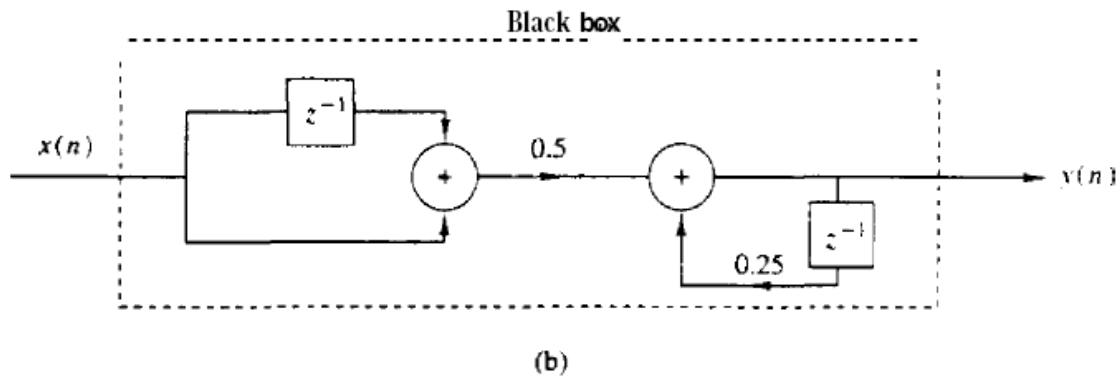
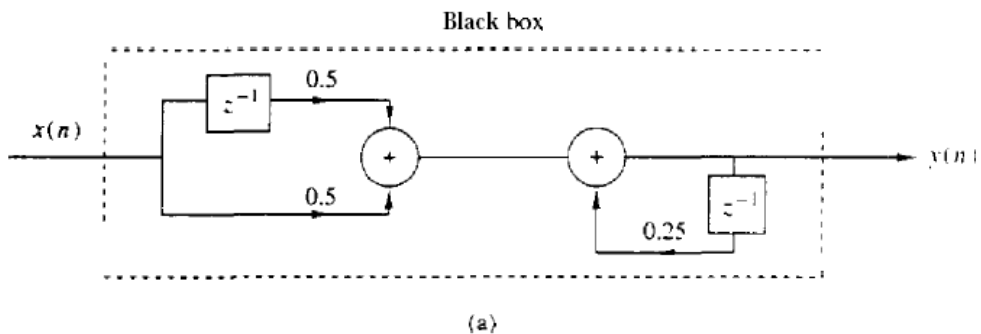
$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$$

where $x(n)$ is the input and $y(n)$ is the output of the system.

Solution : the output $y(n)$ is obtained by multiplying Input $x(n)$ by 0.5, multiplying the previous input $x(n-1)$ by 0.5, adding the two products, and then adding the previous output $y(n-1)$ multiplied by 1/4. Figure a illustrates this block diagram realization of the system. A simple rearrangement .

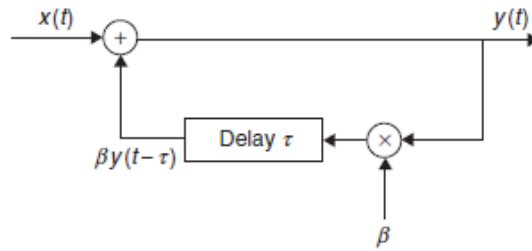
$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}[x(n) + x(n-1)]$$

leads to the block diagram realization shown in Fig. b. Note that if we treat "the system" from the "viewpoint" of an input-output or an external description.



FIGURE

Positive feedback system: the microphone picks up input signal $x(t)$ and the amplified and delayed signal $\beta y(t - \tau)$, making the system unstable.

**Solution**

The input–output equation is

$$y(t) = x(t) + \beta y(t - \tau)$$

If we use this expression to obtain $y(t - \tau)$, we get that

$$y(t - \tau) = x(t - \tau) + \beta y(t - 2\tau)$$

and replacing it in the input–output equation, we get

$$y(t) = x(t) + \beta[x(t - \tau) + \beta y(t - 2\tau)] = x(t) + \beta x(t - \tau) + \beta^2 y(t - 2\tau)$$

Repeating the above scheme, we will obtain the following expression for $y(t)$ in terms of the input

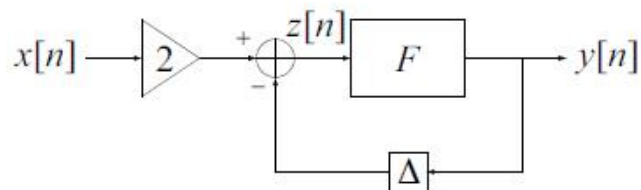
$$y(t) = x(t) + \beta x(t - \tau) + \beta^2 x(t - 2\tau) + \beta^3 x(t - 3\tau) + \dots$$

If we let $x(t) = u(t)$ and $\beta = 2$, the corresponding output is

$$y(t) = u(t) + 2u(t - 1) + 4u(t - 2) + 8u(t - 3) + \dots$$

which continuously grows as time increases. The output is clearly not a bounded signal, although the input is bounded. Thus, the system is unstable, and the screeching sound from the speakers will prove it—you need to separate the speakers and the microphone to avoid it.

2. (15 pts.) Consider the following system:



Here, the system F is defined by the input-output relationship

$$F\{z[n]\} = z[n] - z[n-1],$$

and Δ is the unit delay

$$\Delta\{w[n]\} = w[n-1].$$

Write down the linear difference equation describing this system.

Solution. Let $z[n]$ be the output of the summer, as shown above. Then

$$y[n] = F\{z[n]\} = z[n] - z[n-1].$$

Now,

$$z[n] = 2x[n] - \Delta\{y[n]\} = 2x[n] - y[n-1].$$

Therefore, substituting the expression for $z[n]$ into the first equation, we can write

$$\begin{aligned} y[n] &= z[n] - z[n-1] \\ &= \underbrace{(2x[n] - y[n-1])}_{=z[n]} - \underbrace{(2x[n-1] - y[n-2])}_{=z[n-1]} \\ &= 2x[n] - y[n-1] - 2x[n-1] + y[n-2]. \end{aligned}$$

Simplify to get

$$\boxed{y[n] + y[n-1] - y[n-2] = 2x[n] - 2x[n-1]}$$

2.9. Ayırık Zamanlı Sistemlerin Sınıflandırılması

In the analysis as well as in the design of systems, it is desirable to classify the systems according to the general properties that they satisfy. In fact, the mathematical techniques that we develop in this and in subsequent chapters for analyzing and designing discrete-time systems depend heavily on the general characteristics of the systems that are being considered. For this reason it is necessary for us to develop a number of properties or categories that can be used to describe the general characteristics of systems.

We stress the point that for a system to possess a given property, the property must hold for every possible input signal to the system. If a property holds for some input signals but not for others, the system does not possess that property.

Thus a counterexample is sufficient to prove that a system does not possess a property. However, to prove that the system has some property, we must prove that this property holds for every possible input signal.

Static versus dynamic systems. A discrete-time system is called static or memoryless if its output at any instant n depends at most on the input sample at the same time, but not on past or future samples of the input. In any other case, the system is said to be *dynamic* or to have memory. If the output of a system at time n is completely determined by the input samples in the interval from $n - N$ to n ($N > 0$), the system is said to have *memory* of duration N . If $N = 0$, the system is static. If $0 < N < \infty$, the system is said to have *finite memory*. Whereas if $N = \infty$, the system is said to have *infinite memory*.

The systems described by the following input-output equations

$$y(n) = ax(n)$$

$$y(n) = nx(n) + bx^3(n)$$

are both static or memoryless. Note that there is no need to store any of the past inputs or outputs in order to compute the present output. On the other hand, The systems described by the following input-output relations

$$y(n) = x(n) + 3x(n - 1)$$

$$y(n) = \sum_{k=0}^n x(n - k)$$

$$y(n) = \sum_{k=0}^{\infty} x(n - k)$$

are dynamic systems or systems with memory.

The first two systems described above have finite memory, whereas the last system has infinite memory.

Time-invariant versus time-variant systems. We can subdivide the general class of systems into the two broad categories, time-invariant systems and time-variant systems. A system is called time-invariant if its input-output characteristics do not change with time. To elaborate, suppose that we have a system T in a relaxed state which, when excited by an input signal $x(n)$, produces an output signal $y(n)$. Thus we write

$$y(n) = T[x(n)]$$

Now suppose that the same input signal is delayed by k units of time to yield $x(n - k)$, and again applied to the same system. If the characteristics of the system do not change with time, the output of the relaxed system will be $y(n - k)$. That is, the output will be the same as the response to $x(n)$, except that it will be delayed by the same k units in time that the input was delayed. This leads us to define a time-invariant or shift-invariant system as follows.

Definition. A relaxed system T is **Time invariant** or **shift invariant** if and only if

$$x(n) \xrightarrow{T} y(n)$$

implies that

$$x(n - k) \xrightarrow{T} y(n - k)$$

for every input signal $x(n)$ and every time shift k

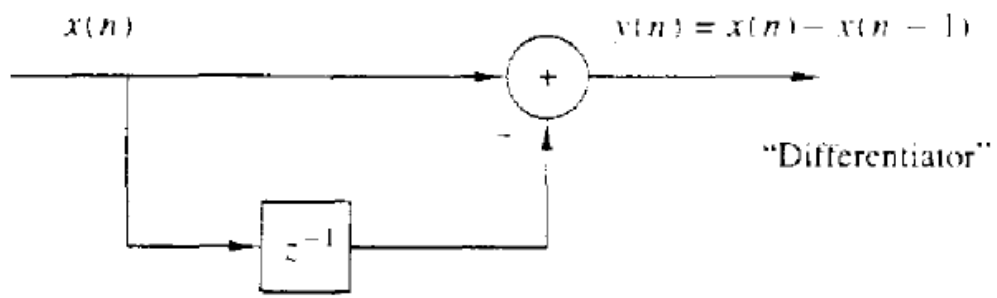
To determine if any given system is time invariant, we need to perform the test specified by the preceding definition. Basically, we excite the system with an arbitrary input sequence $x(n)$, which produces an output denoted as $y(n)$. Next we delay the input sequence by same amount k and recompute the output. In general, we can write the output as

$$y(n, k) = T[x(n - k)]$$

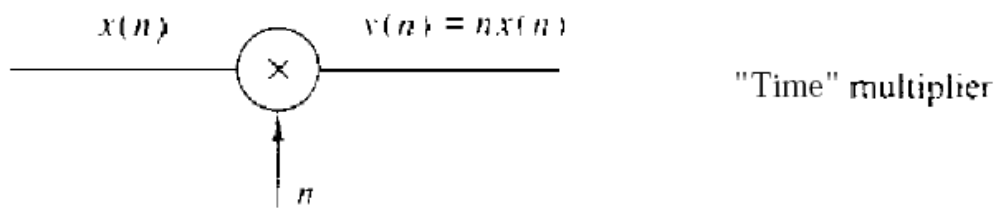
Now if this output $y(n, k) = y(n - k)$, for all possible values of k , the system is time invariant. On the other hand, if the output $y(n, k) \neq y(n - k)$, even for **one** value of k , the system is time variant.

Example .

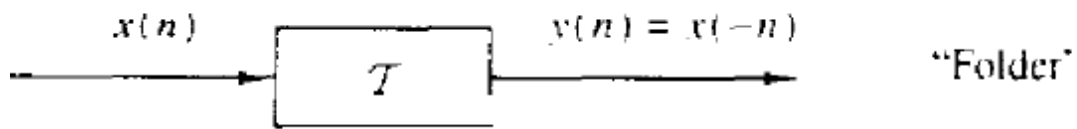
Determine if the systems shown in Fig. below are time invariant or time variant.



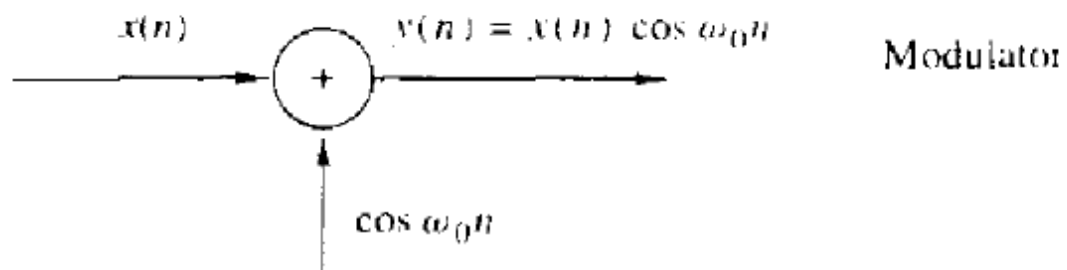
(a)



(b)



(c)



(d)

Solution

(a) This system is described by the input-output equations

$$y(n) = \mathcal{T}[x(n)] = x(n) - x(n - 1)$$

Now if the input is delayed by k units in time and applied to the system, it is clear from the block diagram that the output will be

$$y(n, k) = x(n - k) - x(n - k - 1)$$

On the other hand if we delay $y(n)$ by k units in time, we obtain

$$y(n - k) = x(n - k) - x(n - k - 1)$$

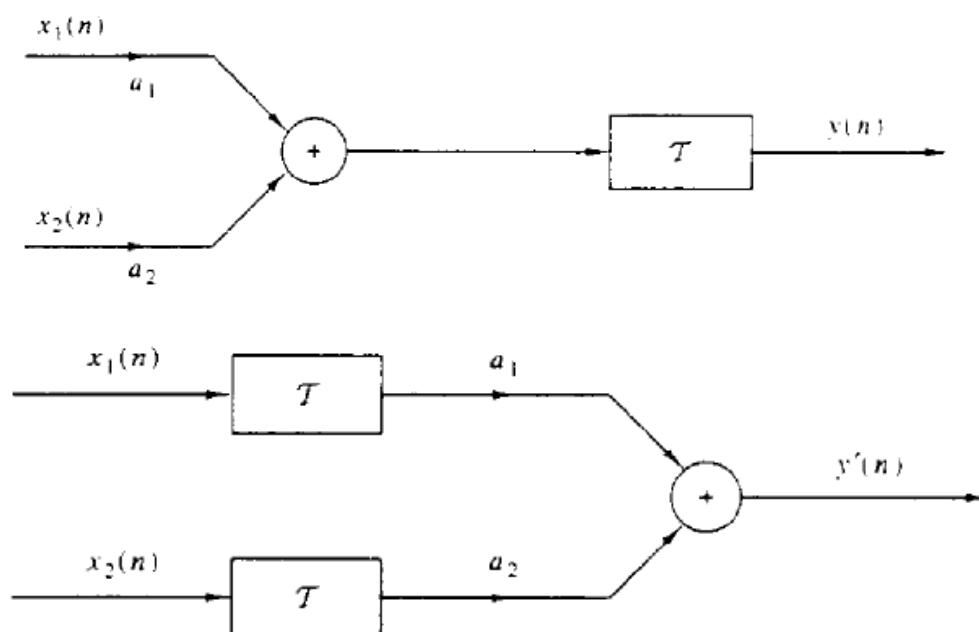
$y(n - k) = y(n, k)$. Therefore, the system is time invariant.

Linear versus nonlinear systems. The general class of systems can also be subdivided into linear systems and nonlinear systems. A linear system is **one** that satisfies the *superposition principle*. Simply stated, the principle of superposition requires that the response of the system to a weighted **sum** of signals be equal to the corresponding weighted sum of the responses (outputs) of the system to each of the individual input signals. Hence we have the following definition of linearity.

Definition. A relaxed \mathcal{T} system is linear if and only if

$$\mathcal{T}[a_1x_1(n) + a_2x_2(n)] = a_1\mathcal{T}[x_1(n)] + a_2\mathcal{T}[x_2(n)]$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$, and any arbitrary constants a_1 and a_2 . Figure below gives a pictorial illustration of the superposition principle.



Determine if the systems described by the following input-output equations are linear or nonlinear.

$$(a) \ y(n) = nx(n) \quad (b) \ y(n) = x(n^2) \quad (c) \ y(n) = x^2(n)$$

$$(d) \ y(n) = Ax(n) + B \quad (e) \ y(n) = e^{x(n)}$$

Solution

(a) For two input sequences $x_1(n)$ and $a_2(n)$, the corresponding outputs are

$$y_1(n) = nx_1(n)$$

$$y_2(n) = nx_2(n)$$

A linear combination of the two input sequences results in the output

$$y_3(n) = T[a_1x_1(n) + a_2x_2(n)] = n[a_1x_1(n) + a_2x_2(n)]$$

$$= a_1nx_1(n) + a_2nx_2(n)$$

On the other hand, a linear combination of the **two** outputs result in the output

$$a_1y_1(n) + a_2y_2(n) = a_1nx_1(n) + a_2nx_2(n)$$

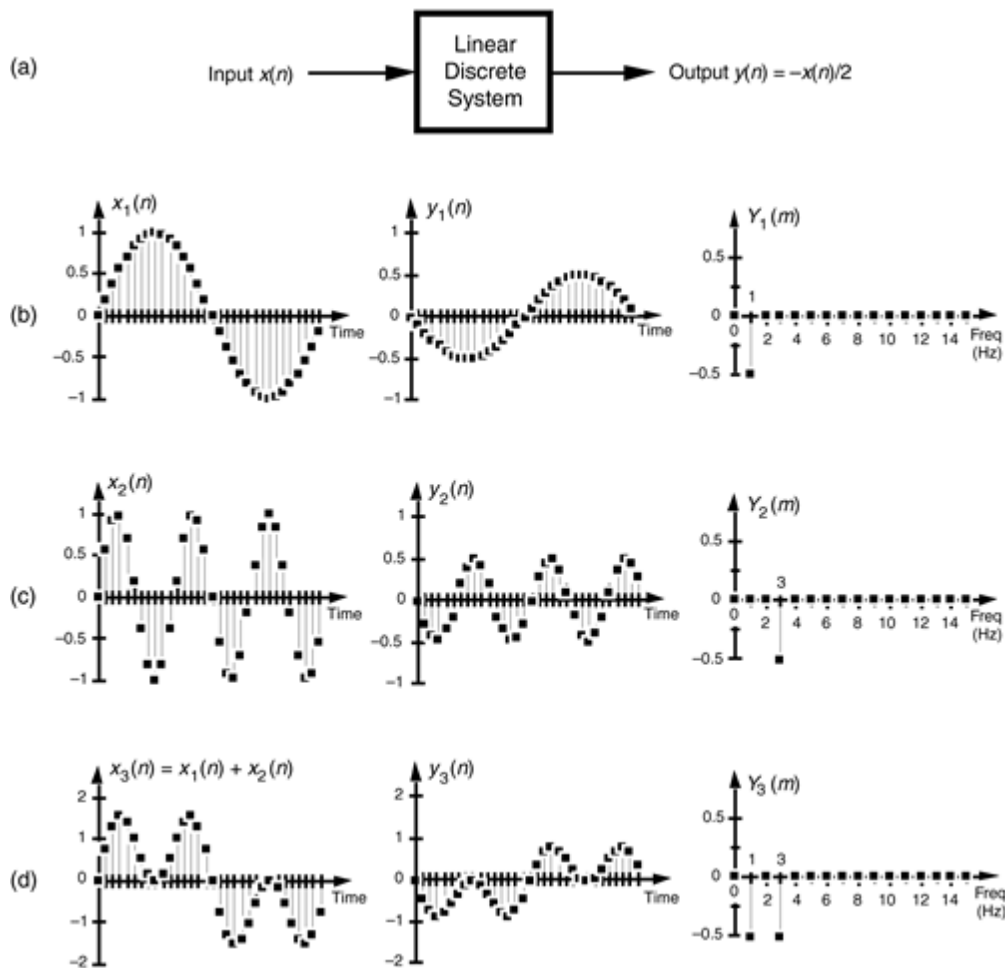
Since they are identical, the system is linear.

Example of a Linear System

To illustrate system linearity, let's say we have the discrete system shown below in Figure (a) whose output is defined as

$$y(n) = \frac{-x(n)}{2},$$

block diagram where $y(n) = -x(n)/2$; (b) system input and output with a 1-Hz sine wave applied; (c) with a 3-Hz sinewave applied; (d) with the sum of 1-Hz and 3-Hz sinewaves applied.



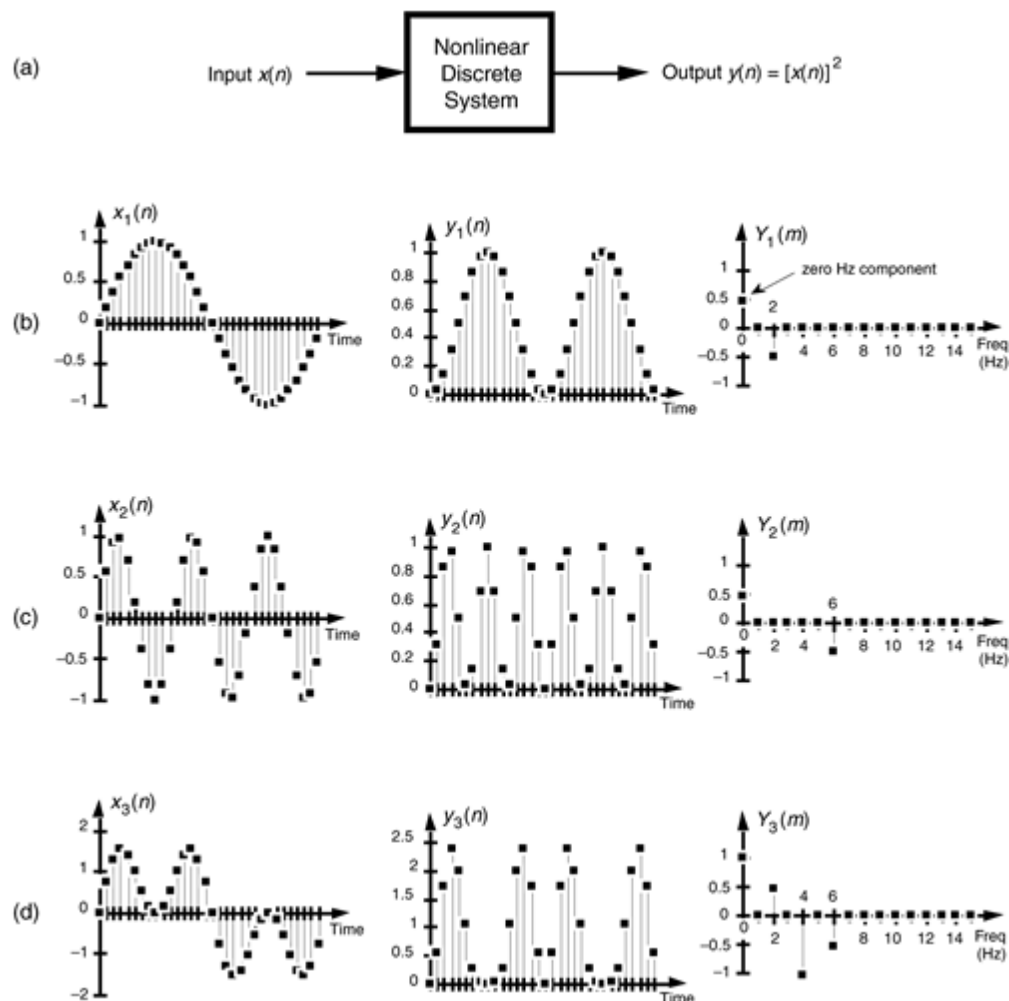
that is, the output sequence is equal to the negative of the input sequence with the amplitude reduced by a factor of two. If we apply an $x_1(n)$ input sequence representing a 1-Hz sinewave sampled at a rate of 32 samples per cycle, we'll have a $y_1(n)$ output as shown in the center of Figure (b). The frequency-domain spectral amplitude of the $y_1(n)$ output is the plot on the right side of Figure (b), indicating that the output comprises a single tone of peak amplitude equal to -0.5 whose frequency is 1 Hz. Next, applying an $x_2(n)$ input sequence representing a 3-Hz sinewave, the system provides a $y_2(n)$ output sequence, as shown in the center of Figure (c). The spectrum of the $y_2(n)$ output, $Y_2(m)$, confirming a single 3-Hz sinewave output is shown on the right side of Figure (c). Finally—here's where the linearity comes in—if we apply an $x_3(n)$ input sequence that's the sum of a 1-Hz sinewave and a 3-Hz sinewave, the $y_3(n)$ output is as shown in the center of Figure (d). Notice how $y_3(n)$ is the sample-for-sample sum of $y_1(n)$ and $y_2(n)$. Figure (d) also shows that the output spectrum $Y_3(m)$ is the sum of $Y_1(m)$ and $Y_2(m)$. That's linearity.

Example of a Nonlinear System

It's easy to demonstrate how a nonlinear system yields an output that is not equal to the sum of $y_1(n)$ and $y_2(n)$ when its input is $x_1(n) + x_2(n)$. A simple example of a nonlinear discrete system is that in shown below in Figure (a) where the output is the square of the input described by

$$y(n) = [x(n)]^2 .$$

Nonlinear system input-to-output relationships: (a) system block diagram where $y(n) = [x(n)]^2$; (b) system input and output with a 1-Hz sinewave applied; (c) with a 3-Hz sinewave applied; (d) with the sum of 1-Hz and 3-Hz sinewaves applied.



We'll use a well known trigonometric identity and a little algebra to predict the output of this nonlinear system when the input comprises simple sinewaves. Let's describe a sinusoidal sequence, whose frequency $f_0 = 1$ Hz, by

$$x_1(n) = \sin(2\pi f_0 n t_s) = \sin(2\pi \cdot 1 \cdot n t_s) .$$

describes the $x_1(n)$ sequence on the left side of Figure (b). Given this $x_1(n)$ input sequence, the $y_1(n)$ output of the nonlinear system is the square of a 1-Hz sinewave, or

$$y_1(n) = [x_1(n)]^2 = \sin(2\pi \cdot 1 \cdot n t_s) \cdot \sin(2\pi \cdot 1 \cdot n t_s) .$$

We can simplify our expression for $y_1(n)$ in by using the following trigonometric identity:

$$\sin(\alpha) \cdot \sin(\beta) = \frac{\cos(\alpha - \beta)}{2} - \frac{\cos(\alpha + \beta)}{2} .$$

we can express $y_1(n)$ as

$$\begin{aligned} y_1(n) &= \frac{\cos(2\pi \cdot 1 \cdot nt_s - 2\pi \cdot 1 \cdot nt_s)}{2} - \frac{\cos(2\pi \cdot 1 \cdot nt_s + 2\pi \cdot 1 \cdot nt_s)}{2} \\ &= \frac{\cos(0)}{2} - \frac{\cos(4\pi \cdot 1 \cdot nt_s)}{2} = \frac{1}{2} - \frac{\cos(2\pi \cdot 2 \cdot nt_s)}{2} , \end{aligned}$$

which is shown as the all positive sequence in the center of Figure (b). Because Equation above results in a frequency sum ($\alpha + \beta$) and frequency difference ($\alpha - \beta$) effect when multiplying two sinusoids, the $y_1(n)$ output sequence will be a cosine wave of 2 Hz and a peak amplitude of -0.5 , added to a constant value of $1/2$. The constant value of $1/2$ in Eq. below is interpreted as a zero Hz frequency component, as shown in the $Y_1(m)$ spectrum in Figure (b). We could go through the same algebraic exercise to determine that, when a 3-Hz sinewave $x_2(n)$ sequence is applied to this nonlinear system, the output $y_2(n)$ would contain a zero Hz component and a 6 Hz component, as shown in Figure 1 (c). System nonlinearity is evident if we apply an $x_3(n)$ sequence comprising the sum of a 1-Hz and a 3-Hz sinewave as shown in Figure (d). We can predict the frequency content of the $y_3(n)$ output sequence by using the algebraic relationship

$$(a+b)^2 = a^2 + 2ab + b^2 ,$$

where a and b represent the 1-Hz and 3-Hz sinewaves, respectively. From Equation:

$$\sin(\alpha) \cdot \sin(\beta) = \frac{\cos(\alpha - \beta)}{2} - \frac{\cos(\alpha + \beta)}{2} .$$

the a^2 term in Equation:

$$(a+b)^2 = a^2 + 2ab + b^2 ,$$

generates the zero-Hz and 2-Hz output sinusoids in Figure (b). Likewise, the b^2 term produces in $y_3(n)$ another zero-Hz and the 6-Hz sinusoid in Figure (c). However, the $2ab$ term yields additional 2-Hz and 4-Hz sinusoids in $y_3(n)$. We can show this algebraically by using Equation:

$$\sin(\alpha) \cdot \sin(\beta) = \frac{\cos(\alpha - \beta)}{2} - \frac{\cos(\alpha + \beta)}{2} .$$

and expressing the $2ab$ term in Equation:

$$(a+b)^2 = a^2 + 2ab + b^2 ,$$

as

$$2ab = 2 \cdot \sin(2\pi \cdot 1 \cdot nt_s) \cdot \sin(2\pi \cdot 3 \cdot nt_s)$$

$$= \frac{2 \cos(2\pi \cdot 1 \cdot nt_s - 2\pi \cdot 3 \cdot nt_s)}{2} - \frac{2 \cos(2\pi \cdot 1 \cdot nt_s + 2\pi \cdot 3 \cdot nt_s)}{2}$$

$$= \cos(2\pi \cdot 2 \cdot nt_s) - \cos(2\pi \cdot 4 \cdot nt_s) .^\dagger$$

Equation above tells us that two additional sinusoidal components will be present in $y_3(n)$ because of the system's nonlinearity, a 2-Hz cosine wave whose amplitude is +1 and a 4-Hz cosine wave having an amplitude of -1. These spectral components are illustrated in $Y_3(m)$ on the right side of Figure (d).

Notice that, when the sum of the two sinewaves is applied to the nonlinear system, the output contained sinusoids, Equation above, that were not present in either of the outputs when the individual sinewaves alone were applied. Those extra sinusoids were generated by an interaction of the two input sinusoids due to the squaring operation. That's nonlinearity. Although nonlinear systems are usually difficult to analyze, they are occasionally used in practice, for example, describe their application in nonlinear digital filters.

Causal versus noncausal systems. We begin with the definition of causal discrete-time systems.

Definition. A system is said to be causal if the output of the system at any time n [i.e., $y(n)$] depends only on present and past inputs [i.e., $x(n), x(n-1), \dots, x(n-2), \dots$], but does not depend on future inputs [i.e., $x(n+1), x(n+2), \dots$]. In mathematical terms, the output of a causal system satisfies an equation of the form

$$y(n) = F[x(n), x(n-1), x(n-2), \dots]$$

where $F[\cdot]$ is some arbitrary function.

If a system does not satisfy this definition, it is called **noncausal**. Such a system has an output that depends not only on present and past inputs but also on future inputs. It is apparent that in real-time signal processing applications we cannot observe future values of the signal, and hence a noncausal system is physically unrealizable (i.e., it cannot be implemented). On the other hand, if the signal is recorded so that the processing is done off-line (nonreal time), it is possible to implement a noncausal system, since all values of the signal are available at the time of processing.

This is often the case in the processing of geophysical signals and images.

Example .

Determine if the systems described by the following input-output equations are causal or noncausal.

$$(a) y(n) = x(n) - x(n-1) \quad (b) y(n) = \sum_{k=-\infty}^n x(k) \quad (c) y(n) = ax(n)$$

$$(d) y(n) = x(n) + 3x(n+4) \quad (e) y(n) = x(n^2) \quad (f)$$

$$y(n) = x(-n)$$

Stable versus unstable systems. Stability is an important property that must be considered in any practical application of a system. Unstable systems usually exhibit erratic and extreme behavior and cause overflow in any practical implementation.

Definition: A system is said to be stable in the bounded input-bounded output sense if, for any input that is bounded,

$$|x(n)| \leq A < \infty,$$

the output will be bounded,

$$|y(n)| \leq B < \infty$$

For a linear shift-invariant system, stability is guaranteed if the unit sample response is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \quad (1.8)$$

EXAMPLE An LSI system with unit sample response $h(n) = a^n u(n)$ will be stable whenever $|a| < 1$, because

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} \quad |a| < 1$$

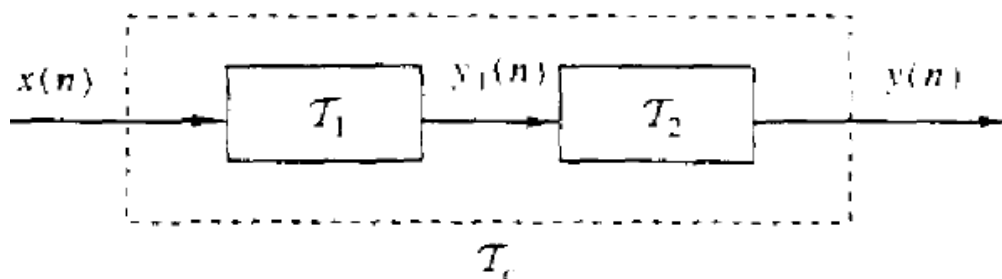
The system described by the equation $y(n) = nx(n)$, on the other hand, is not stable because the response to a unit step, $x(n) = u(n)$, is $y(n) = nu(n)$, which is unbounded.

Interconnection of Discrete-Time Systems

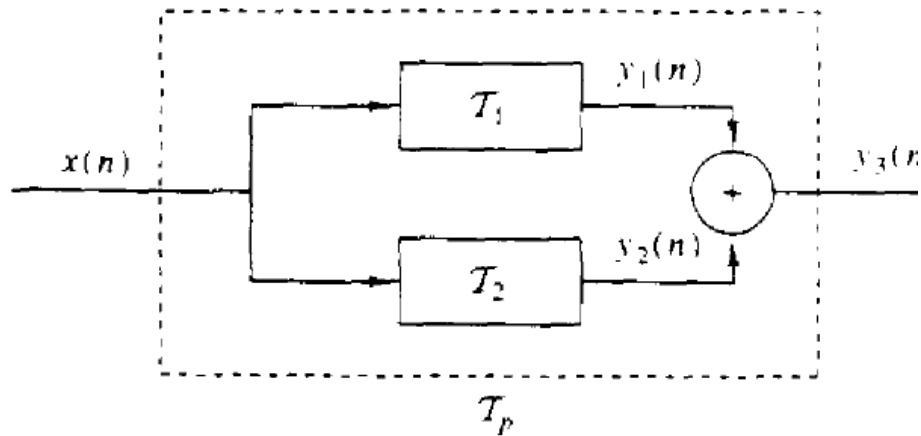
Discrete-time systems can be interconnected to form larger systems. There are two basic ways in which systems can be interconnected: in cascade (series) or in parallel. These interconnections are illustrated in Fig. below. Note that the two interconnected systems are different.

In the cascade interconnection the output of the first system is

$$y_1(n) = \mathcal{T}_1\{x(n)\}$$



(a)



(b)

and the output of the second system is

$$\begin{aligned}
 y(n) &= T_2[y_1(n)] \\
 &= T_2\{T_1[x(n)]\}
 \end{aligned}$$

We observe that systems T_1 and T_2 can be combined or consolidated into a single overall system

$$T_c \equiv T_2 T_1$$

Consequently, we can express the output of the combined system as

$$y(n) = T_c[x(n)]$$

In general, the order in which the operations T_1 and T_2 are performed is important. That is,

$$T_2 T_1 \neq T_1 T_2$$

for arbitrary systems. However, if the systems T_1 and T_2 are linear and time invariant, then (a) T_c is time invariant and (b) $T_2 T_1 = T_1 T_2$ that is, the order in which the systems process the signal is not important. $T_2 T_1$ and $T_1 T_2$ yield identical output sequences, In the parallel interconnection, the output of the system T_1 is $y_1(n)$ and the output of the system T_2 is $y_2(n)$. Hence the output of the parallel interconnection is

$$\begin{aligned}y_3(n) &= y_1(n) + y_2(n) \\ &= \mathcal{T}_1[x(n)] + \mathcal{T}_2[x(n)] \\ &= (\mathcal{T}_1 + \mathcal{T}_2)[x(n)] \\ &= \mathcal{T}_p[x(n)]\end{aligned}$$

$$\mathcal{T}_p = \mathcal{T}_1 + \mathcal{T}_2.$$

In general, we can use parallel and cascade interconnection of systems to construct larger, more complex systems. Conversely, we can take a larger system and break it down into smaller subsystems for purposes of analysis and implementation. We shall use these notions later, in the design and implementation of digital filters.

3. Fourier Transform

Fourier dönüşümü, sadece sinüs ve kosinüs temel işlevleri kullanılarak sentezlenebilen sinyalleri analiz etmek için yeterlidir.

Dünyadaki birçok şey, bir dalga formu aracılığıyla tanımlanabilir - zaman, mekan veya başka bir değişkenin fonksiyonu. Örneğin, ses dalgaları, elektromanyetik alanlar, radyodan dinlediğiniz müzik, hisse senetlerini zamana göre fiyatı, nefesinizin sıklığı vb.

Fourier Dönüşümü, bize bu dalga formlarını doğrudan görüntülemenin benzersiz ve güçlü bir yolunu sunduğu için önemli bir rol oynamaktadır. Fourier Transform, görüntü analizi, görüntü filtreleme, görüntü rekonstrüksiyonu ve görüntü sıkıştırması gibi çok çeşitli uygulamalarda kullanılır.

Bir Fransız matematikçi ve fizikçi Jean Baptiste Joseph Fourier, Fourier analizini geliştirdi. Periyodik sinyalin uygun seçilmiş sinüzoidal dalgaların toplamı olarak temsil edilebileceği konusunda tartışmalı bir iddiaya sahipti. Bu yazının bir gözden geçircisi olan matematikçi Lagrange, süresiz eğimler gibi köşeleri olan sinyalleri temsil etmek için bir yaklaşımın kullanılamayacağı konusunda ısrar etti. Lagrange'ın görüşü doğrudu ama tam olarak değil, çünkü sıfır enerjiye sahip iki sinüzoidal işaret arasındaki fark çok yakındı. Makale sonunda Lagrange öldükten sonra yayınlandı. Fourier'in genelleme iddiasının biraz kuvvetli olduğu ortaya çıksa da, sonuçları günümüze kadar devam eden önemli bir araştırma selini harekete geçirdi.

Fourier dönüşümü fizik ve mühendislik alanındaki birçok uygulama ile matematiksel bir dönüşümdür. Fourier serileri denilen trigonometrik serileri kullanarak kısmi diferansiyel denklemleri içeren birçok önemli problemi çözebiliriz.

Evrende gözlediğiniz tüm dalga formları farklı frekans ve genliklere sahip sinüs fonksiyonlarının toplamından ibarettir!

The Fourier transform

we'll be interested in signals defined for all t

the **Fourier transform** of a signal f is the function

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$$

- F is a function of a *real* variable ω ; the function value $F(\omega)$ is (in general) a complex number

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t dt - j \int_{-\infty}^{\infty} f(t) \sin \omega t dt$$

- $|F(\omega)|$ is called the *amplitude spectrum* of f ; $\angle F(\omega)$ is the *phase spectrum* of f
- notation: $F = \mathcal{F}(f)$ means F is the Fourier transform of f ; as for Laplace transforms we usually use uppercase letters for the transforms (e.g., $x(t)$ and $X(\omega)$, $h(t)$ and $H(\omega)$, etc.)

Fourier transform and Laplace transform

Laplace transform of f

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

Fourier transform of f

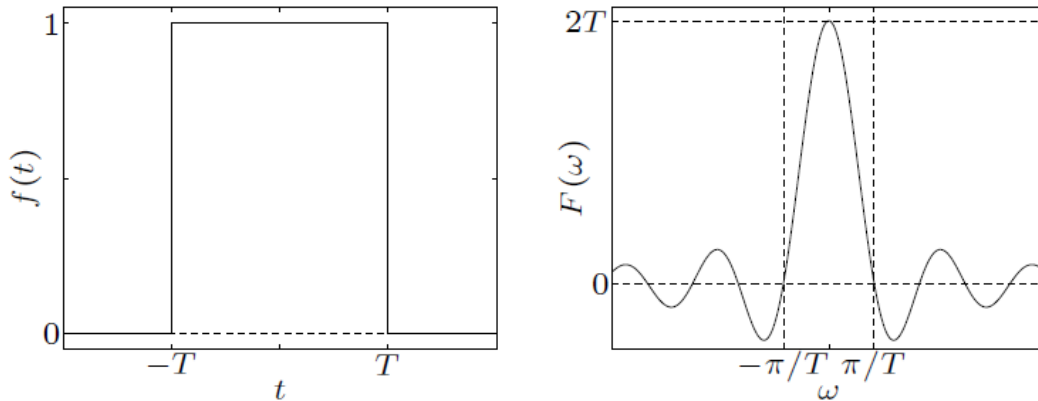
$$G(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$$

very similar definitions, with two differences:

- Laplace transform integral is over $0 \leq t < \infty$; Fourier transform integral is over $-\infty < t < \infty$
- Laplace transform: s can be any complex number in the region of convergence (ROC); Fourier transform: $j\omega$ lies on the imaginary axis

rectangular pulse: $f(t) = \begin{cases} 1 & -T \leq t \leq T \\ 0 & |t| > T \end{cases}$

$$F(\omega) = \int_{-T}^T e^{-j\omega t} dt = \frac{-1}{j\omega} (e^{-j\omega T} - e^{j\omega T}) = \frac{2 \sin \omega T}{\omega}$$



unit impulse: $f(t) = \delta(t)$

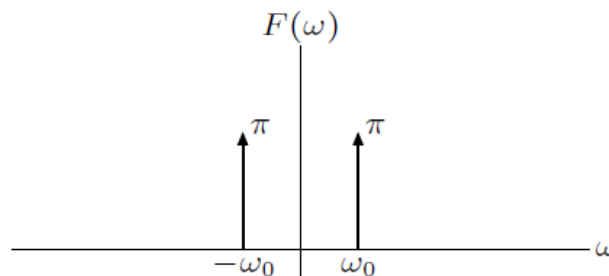
$$F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Fourier transform of periodic signals

similarly, by allowing impulses in $\mathcal{F}(f)$, we can define the Fourier transform of a periodic signal

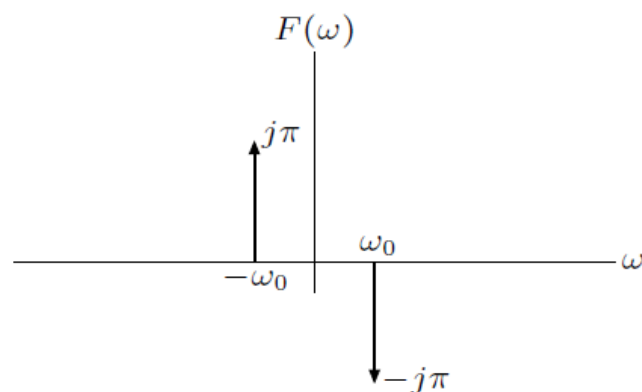
sinusoidal signals: Fourier transform of $f(t) = \cos \omega_0 t$

$$\begin{aligned} F(\omega) &= \frac{1}{2} \int_{-\infty}^{\infty} (e^{j\omega_0 t} + e^{-j\omega_0 t}) e^{-j\omega t} dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} e^{-j(\omega - \omega_0)t} dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j(\omega + \omega_0)t} dt \\ &= \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0) \end{aligned}$$



Fourier transform of $f(t) = \sin \omega_0 t$

$$\begin{aligned}
 F(\omega) &= \frac{1}{2j} \int_{-\infty}^{\infty} (e^{j\omega_0 t} - e^{-j\omega_0 t}) e^{-j\omega t} dt \\
 &= \frac{1}{2j} \int_{-\infty}^{\infty} e^{-j(\omega - \omega_0)t} dt + -\frac{1}{2j} \int_{-\infty}^{\infty} e^{-j(\omega_0 + \omega)t} dt \\
 &= -j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)
 \end{aligned}$$



Examples

sign function: $f(t) = \begin{cases} 1 & t \geq 0 \\ -1 & t < 0 \end{cases}$

write f as $f(t) = -1 + 2g(t)$, where g is a unit step at $t = 0$, and apply linearity

$$F(\omega) = -2\pi\delta(\omega) + 2\pi\delta(\omega) + \frac{2}{j\omega} = \frac{2}{j\omega}$$

sinusoidal signal: $f(t) = \cos(\omega_0 t + \phi)$

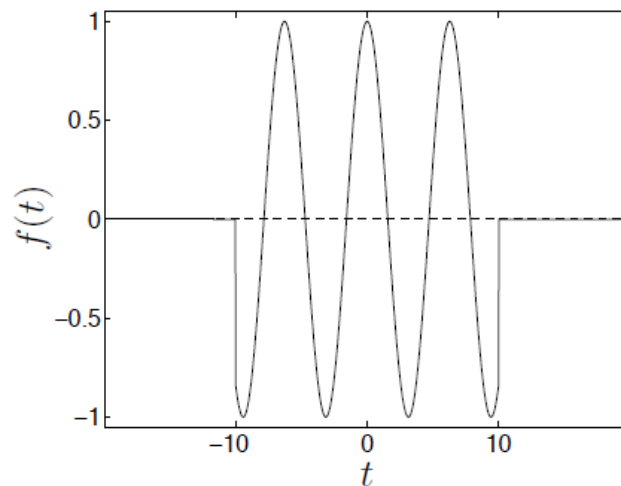
write f as

$$f(t) = \cos(\omega_0(t + \phi/\omega_0))$$

and apply time shift property:

$$F(\omega) = \pi e^{j\omega\phi/\omega_0} (\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$$

pulsed cosine: $f(t) = \begin{cases} 0 & |t| > 10 \\ \cos t & -10 \leq t \leq 10 \end{cases}$



write f as a product $f(t) = g(t) \cos t$ where g is a rectangular pulse of width 20 (see page 12-7)

$$\mathcal{F}(\cos t) = \pi\delta(\omega - 1) + \pi\delta(\omega + 1), \quad \mathcal{F}(g(t)) = \frac{2 \sin 10\omega}{\omega}$$

The inverse Fourier transform

if $F(\omega)$ is the Fourier transform of $f(t)$, *i.e.*,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$$

then

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega$$

let's check

$$\begin{aligned} \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega &= \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \left(\int_{\tau=-\infty}^{\infty} f(\tau)e^{-j\omega\tau} \right) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{\tau=-\infty}^{\infty} f(\tau) \left(\int_{\omega=-\infty}^{\infty} e^{-j\omega(\tau-t)} d\omega \right) d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)\delta(\tau - t) d\tau \\ &= f(t) \end{aligned}$$

Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$$

$$\int e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t}$$

1. Find the Fourier series for (periodic extension of)

$$f(t) = \begin{cases} 1, & t \in [0, 2); \\ -1, & t \in [2, 4). \end{cases}$$

Determine the sum of this series.

2. Find the Fourier series for (periodic extension of)

$$f(t) = \begin{cases} t - 1, & t \in [0, 2); \\ 3 - t, & t \in [2, 4). \end{cases}$$

Determine the sum of this series.

3. Find the sine Fourier series for (periodic extension of)

$$f(t) = \begin{cases} t - 1, & t \in [0, 2); \\ 3 - t, & t \in [2, 4). \end{cases}$$

Determine the sum of this series.

4. Find the cosine Fourier series for (periodic extension of)

$$f(t) = \begin{cases} 1, & t \in [0, 1); \\ 0, & t \in [1, 4). \end{cases}$$

Determine the sum of this series.

5. Find the Fourier series for (periodic extension of)

$$f(t) = 1 - t^2, t \in [-1, 1).$$

Determine the sum of this series.

Soru:

The Fourier series for the function $f(x) = \sin^2 x$ is

$$f(x) = \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$= 0.5 - 0.5 \cos 2x$$

$$f(x) = A_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 x + b_n \sin n\omega_0 x$$

$f(x) = \sin^2 x$ is an even function so $b_n = 0$

$$A_0 = 0.5$$

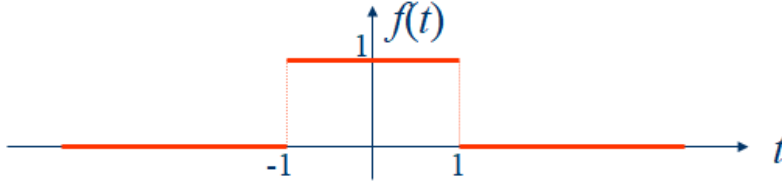
$$a_n = \begin{cases} -0.5, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{T} = 2$$

Soru-15:

Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$$



Şekilde verilmiş olan $f(t)$ 'nin fourier dönüşümünü bulunuz.

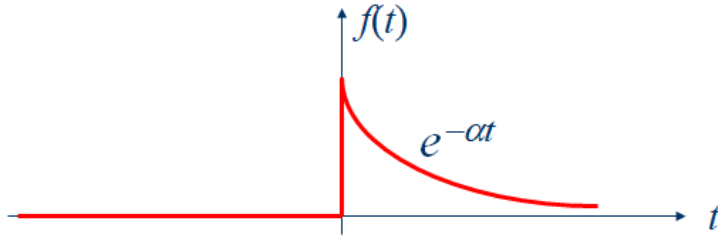
$$\int_{-1}^1 e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-1}^1$$

$$\begin{aligned} F(j\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt = \int_{-1}^1 e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-1}^1 \\ &= \frac{j}{\omega} (e^{-j\omega} - e^{j\omega}) = \frac{2 \sin \omega}{\omega} \end{aligned}$$

Soru-16:

Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$$



Şekilde verilmiş olan $f(t)$ 'nin fourier dönüşümünü bulunuz.

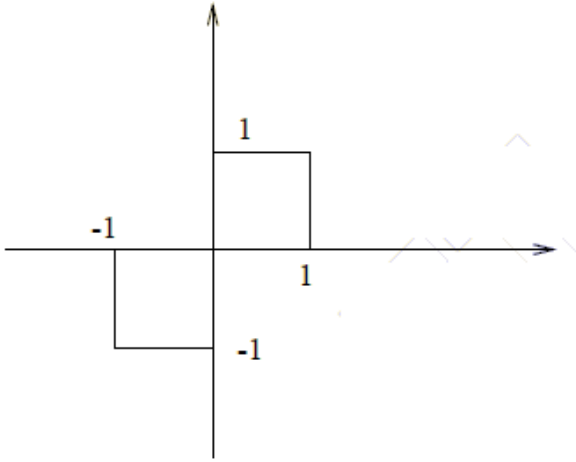
$$\int e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t}$$

$$\begin{aligned} F(j\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt = \int_0^{\infty} e^{-\alpha t} e^{-j\omega t} dt \\ &= \int_0^{\infty} e^{-(\alpha+j\omega)t} dt = \frac{1}{\alpha + j\omega} \end{aligned}$$

Soru-3:

$$x(t) = \begin{cases} -1 & -1 \leq t < 0 \\ 1 & 0 \leq t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

a) $X(t)$ fonksiyonunu çiziniz



b) Fourier dönüşümünü bulunuz

$$\begin{aligned} X(\omega) &= -\int_{-1}^0 e^{-j\omega t} dt + \int_0^1 e^{-j\omega t} dt \\ &= -\int_0^1 e^{j\omega t} dt + \int_0^1 e^{-j\omega t} dt \\ &= -2j \int_0^1 \sin(\omega t) dt \\ &= 2j \frac{1}{\omega} \cos(\omega t) \Big|_0^1 \\ &= 2j \frac{1}{\omega} (\cos(\omega) - 1) \end{aligned}$$

3.1. Matlab ile Sinyal Analizi

$Y = \text{fft}(X)$ computes the discrete Fourier transform (DFT) of X using a fast Fourier transform (FFT) algorithm.

$Y = \text{fft}(X,n)$ returns the n -point DFT. If no value is specified, Y is the same size as X .

Örnek:

Gürültülü Sinyal

Use Fourier transforms to find the frequency components of a signal buried in noise. Specify the parameters of a signal with a sampling frequency of 1000Hz and a signal duration of 150 ms.

```
clear all
close all
Fs = 1000;      % Sampling frequency
T = 1/Fs;      % Sampling period
L = 150;       % Length of signal
t = (0:L-1)*T; % Time vector
```

```
S1 = 0.7*sin(2*pi*50*t);
```

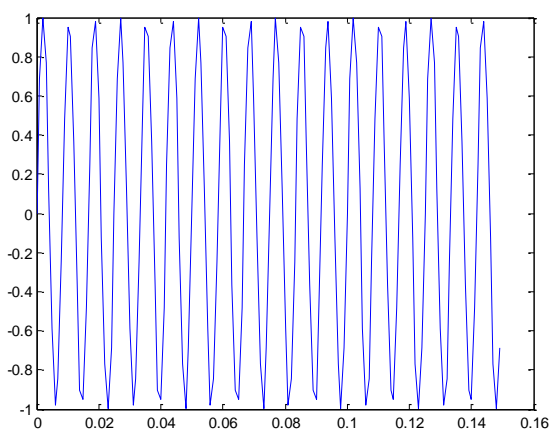
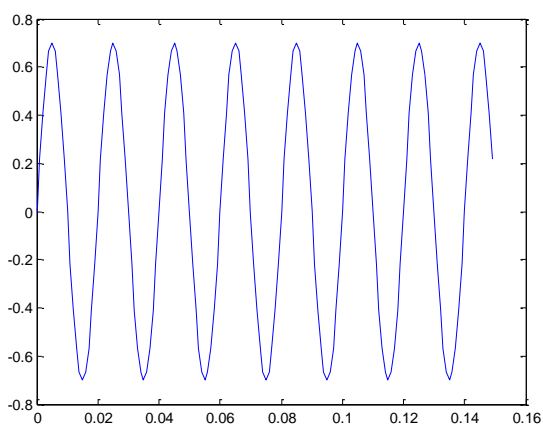
```
S2 = sin(2*pi*120*t);
```

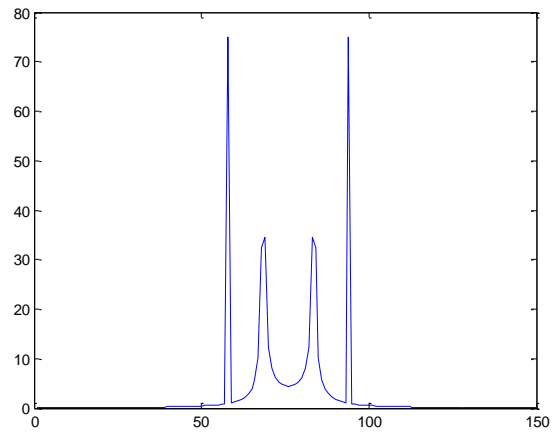
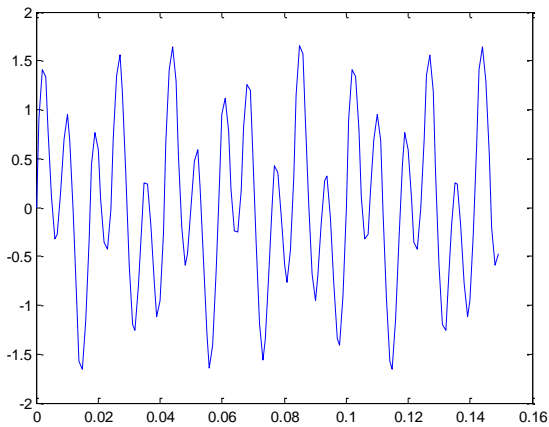
```
figure, plot(t,S1)
```

```
figure, plot(t,S2)
```

```
S=S1+S2;
```

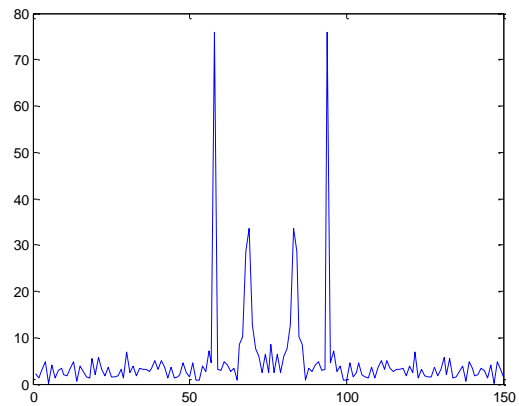
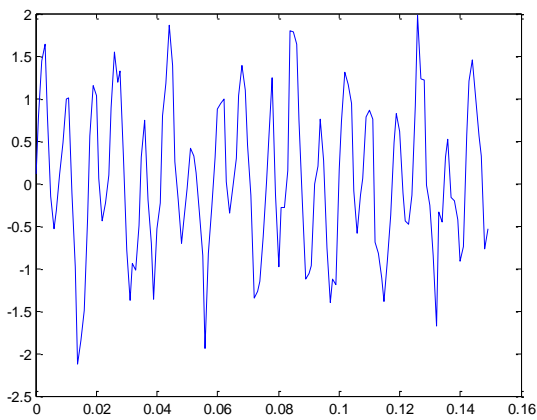
```
figure, plot(t,S)
```



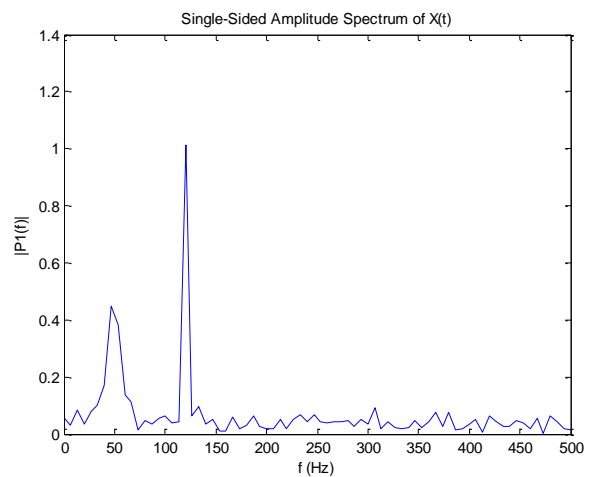


```
Y = fft(S);
figure, plot(fftshift(abs(Y)))
```

```
G = S + 0.25*randn(size(t));
figure, plot(t,G)
FG=fft(G)
figure, plot(fftshift(abs(FG)))
```



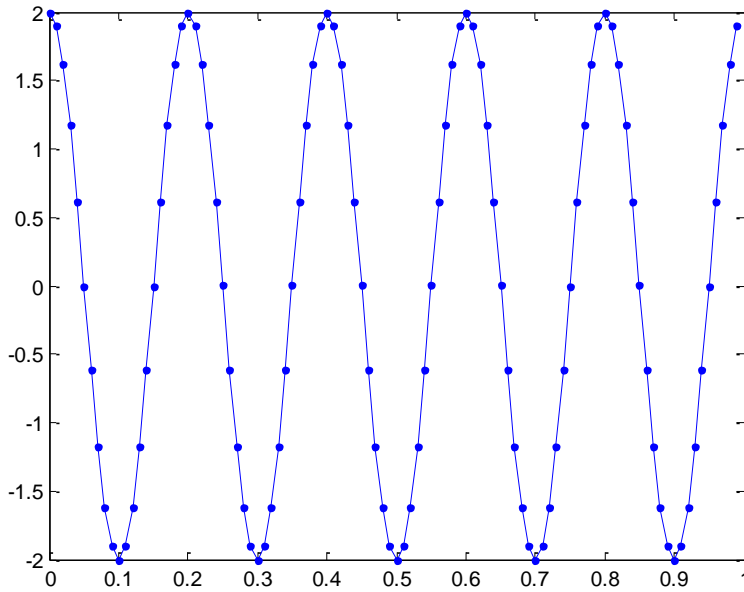
```
P2 = abs(FG/L);
P1 = P2(1:L/2+1);
P1(2:end-1) = 2*P1(2:end-1);
f = Fs*(0:(L/2))/L;
figure, plot(f,P1)
title('Single-Sided Amplitude Spectrum of G(t)')
xlabel('f (Hz)')
ylabel('|P1(f)|')
```



3.2. Fourier analysis and Matlab

Başlangıç olarak, bazı basit 1D sinyalleri oluşturalım ve Fourier dönüşümlerini inceleyelim. Bu ilk örnekte, basit bir periyodik sinyalin frekans içeriğini kontrol edeceğiz. Verilen büyüklük, frekans, örnekleme oranı ve süreye sahip bir kosinüs sinyali oluşturun:

```
clear all
close all
mag = 2;      % magnitude (arbitrary units)
f = 5;       % frequency in Hz
Ps=50;      % number of sampling on a periot
samp = f*Ps; % sampling rate in Hz
t      = 0:1/samp:1-1/samp; % time (1s of data)
N      = length(t)
x = mag*cos(2*pi*f*t); % the signal equation
figure
plot(t,x, 'l.-');
```



Örnekleme hızı (saniyedeki örnek sayısı) ile sinyalin frekansı arasındaki ilişkinin iyi anlaşılması gerekir. Süresi 1 saniye olan bir sinyal üretildiğinden, saniyedeki döngü sayısı kolayca hesaplanabilir:

Frekans=5Hz ise bir sinyalin periyodu= $T=1/5=0.2$ sec.

1 saniyede üretilecek sinyal sayısı=1 saniye * frekans=5 adettir.

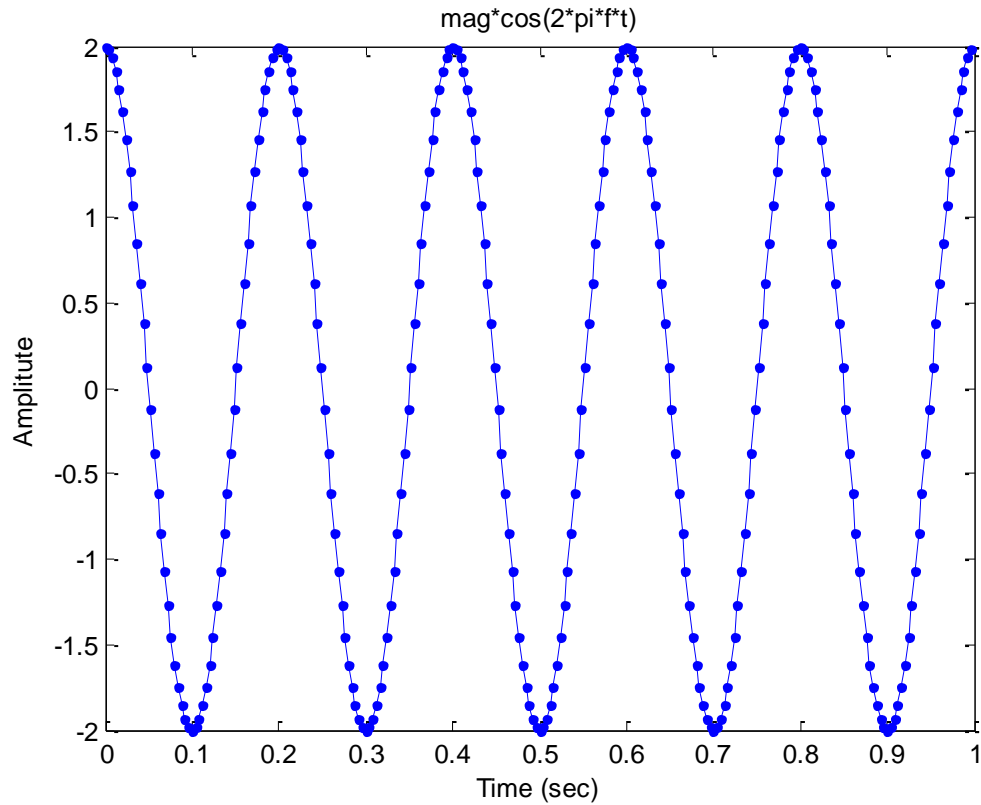
Bir periyotluk sinyaldeki örnek sayısı=toplam örnek sayısı/sinyal sayısı=100/5=20 adettir.

Örnek:

1 saniyede üretilecek sinyal sayısı=2 adet ise $T=1/2=0.5\text{sec}$, $f=1/T=2\text{Hz}$,

Toplam örnek sayısı=Bir periyotluk sinyaldeki örnek sayısı * sinyal sayısı= $40*2=80$ adettir.

```
clear all
close all
mag = 2; % magnitude (arbitrary units)
f = 5; % frequency in Hz
Ps=50; % number of sampling on a periot
samp = f*Ps; % sampling rate in Hz
t = 0:1/samp:1-1/samp; % time (1s of data)
N = length(t)
x = mag*cos(2*pi*f*t); % the signal equation
figure
plot(t,x,'.-');
xlabel('Time (sec)');
ylabel('Amplitude');
title('mag*cos(2*pi*f*t)')
```

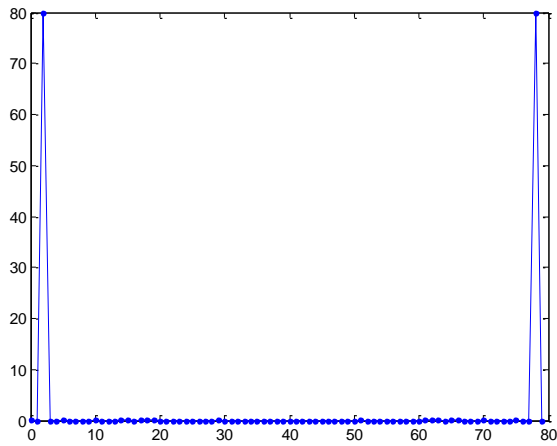


Son zaman noktasının kaldırılma nedeni, FFT yaparken, matlab, sinyali periyodik varsayar ve kendisini süresiz olarak tekrarlar. Matlab'ın sinyalin saf sonsuz bir kosinüs fonksiyonu olduğunu görmesi gerekir.

Bu sinyalin Fourier dönüşümünü alalım,

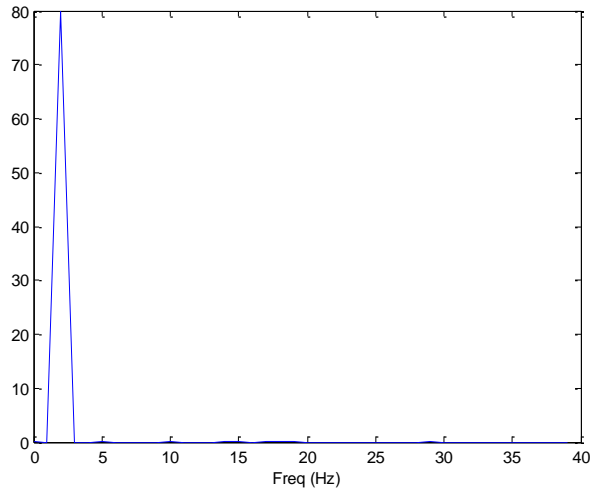
```
clear all
close all
mag = 2;      % magnitude (arbitrary units)
freq = 2;     % frequency in Hz
samp = 80;   % sampling rate in Hz
t      = 0:1/samp:1-1/samp; % time (1s of data)
N      = length(t)
x = mag*cos(2*pi*freq*t); % the signal equation
figure, plot(t,x,'.-');

y = fft(x); % do Fast Fourier Transform
f = linspace(0,N-1,N); % vector of frequencies for plotting
figure, plot(f,abs(y),'.-'); % plotting the magnitude of the FFT
figure, plot(f(1:N/2),abs(y(1:N/2))); % plotting half of the fft results
xlabel('Freq (Hz)'); % labelling x-axis
```



Bu çizime **güç spektrumu** denir.

İlk yükselti kosinüs fonksiyonunun frekansı ile çakışır. İkinci zirve simetriğidir. Bu nedenle, fourier dönüşümünün sadece ilk yarısını gösterilir:



Matlab'da (ifft) ters FFT işlevi:

Biri büyüklük ve diğeri faz olan iki sinyalin ters fourier dönüşümü

Note: Compute the value of $e^{i\pi}$. The notation 1i is Matlab's code for the famous imaginary number $\sqrt{-1}$.

$Y = \exp(1i*\pi)$

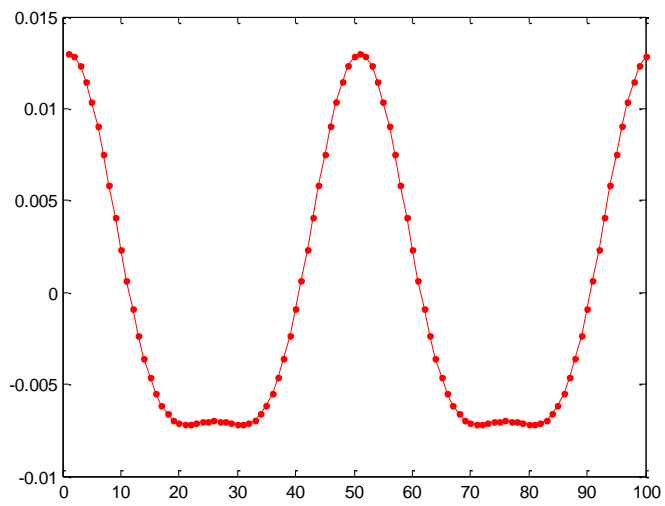
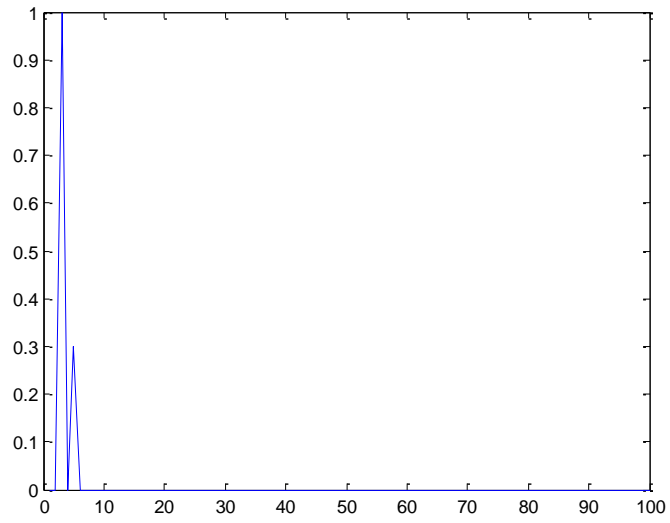
$Y = -1.0000 + 0.0000i$

```
clear all
close all
mag = zeros(1,100);
mag(3)=1;           % spike in the magnitude at freq=3
mag(5)=.3;         % spike in the magnitude at freq=5
ph=zeros(1,100); % zero phase throughout
y = mag.*exp(1i*ph); % the complex signal (fft of some real signal)
figure, plot(abs(y))

x = ifft(y);           % the inverse fft (x is in general complex too)
x1 = real(x); % here we imagine that we can "measure" the real part of x

z = ifft(mag);         % here we just use the magnitude to do the ifft
z1 = real(z);

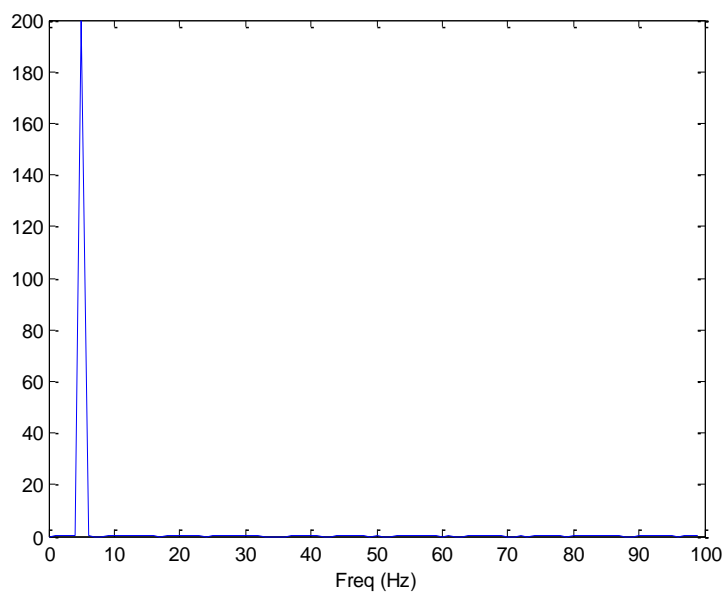
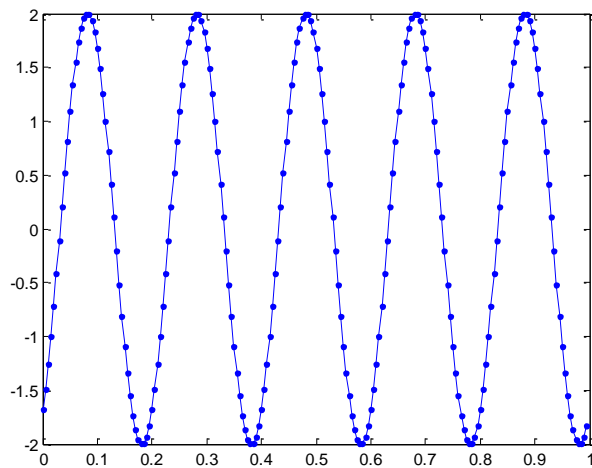
figure, plot(x1, '-r') % plot the real part of the signal
figure, plot(z1, '-b') % plot the ifft of the magnitude
```



Inverse FFT

```
clear all
close all
mag = 2;           % magnitude (arbitrary units)
freq = 5;          % frequency in Hz
samp = 200;       % sampling rate in Hz

t = 0:1/samp:1-1/samp; % time (1s of data)
N = length(t);      % store the number of time points
x = mag*cos(2*pi*freq*t + 10); % the signal equation (+phase)
figure, plot(t,x,'.-');
y = fft(x);         % do Fast Fourier Transform
f = linspace(0,N-1,N); % vector of frequencies for plotting
figure, plot(f(1:N/2),abs(y(1:N/2))); % plotting half of the fft results
xlabel('Freq (Hz)'); % labelling x-axis
z = ifft(y);
plot(t,z,'r--');
```



2D Fourier transform:

Matlab'daki FFT'nin 2 boyutlu versiyonuna FFT2 denir. Şimdi bir resme 2D Fourier dönüşümünü uygulayacağız.

```
clear all
close all
im=imread('Apricot.png');
figure, imshow(im)
im1=im(:,:,1);
figure, imshow(im1)

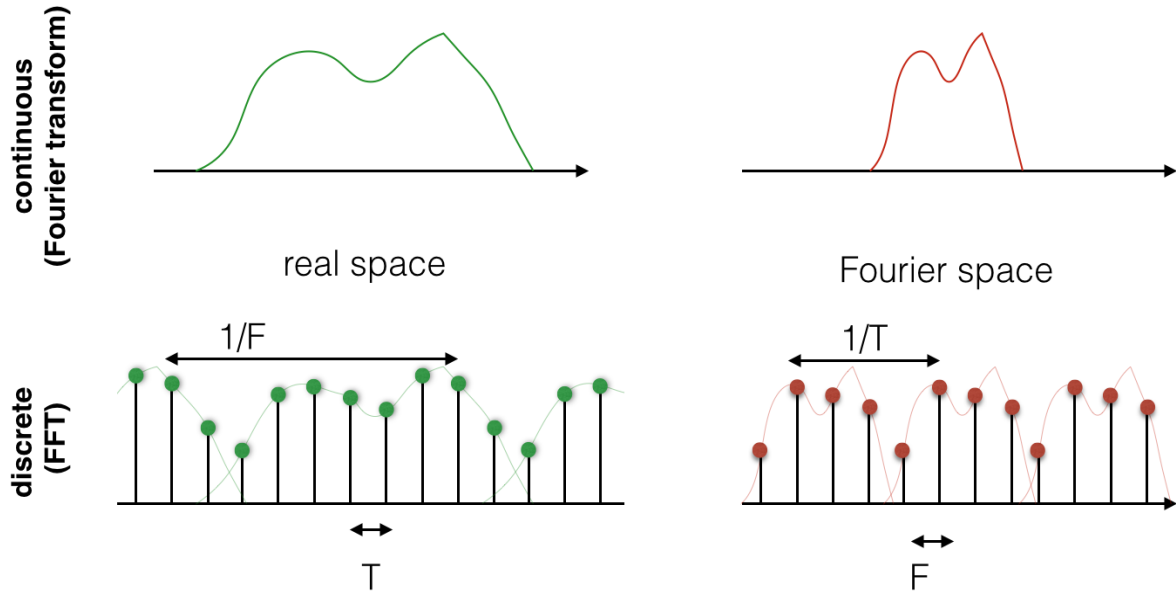
y=fft2(im1);

clim=quantile(abs(y(:)), [.01 .99]);
figure
imagesc(fftshift(abs(y)),clim);colormap gray
title('magnitude');

clim=quantile(angle(y(:)), [.01 .99]);
figure
imagesc(fftshift(angle(y)),clim);colormap gray
title('phase')
```

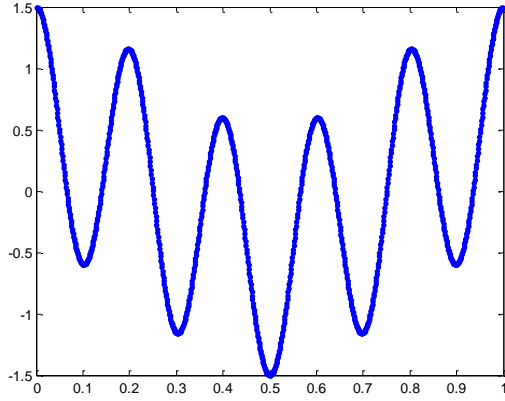
Sampling and aliasing

Verilerin ayrık ya da kesiktir. Örnekleme, gerçek uzayda Fourier dönüşümünü periyodik olarak tekrarlamakla aynıdır ve tekrar süresi, örnekleme süresinin tersidir.



İki kosinüs fonksiyonunun süperpozisyonundan oluşan basit bir sinyali ele alalım. Bu sinyali örnekleme hızında (sürekli gibi) simüle edelim.

```
clear all
close all
freq1 = 5;      % freq in Hz
freq2 = 1;     % freq in Hz
samp  = 1000;  % sampling rate in Hz
t      = 0:1/samp:1; % time (1s of data)
t      = t(1:end-1); % remove last time point
N      = length(t); % store the number of time points
x      = cos(2*pi*freq1*t) + .5*cos(2*pi*freq2*t); % the signal equation
figure
plot(t,x,'.-');
```



Bu sinyal matlab tarafından periyodik olduğu varsayılır, bu nedenle sınırlı bir süreye sahip olsa da, FFT tarafından tekrarlanıyormuş gibi ve dolayısıyla gerçek bir kosinüs fonksiyonu gibi muamele görülür. Bu sinyalden alt örnekler aldığımızda ne olacağını görelim.

```
clear all
close all
freq1 = 5;      % freq in Hz
freq2 = 1;     % freq in Hz
samp  = 1000;  % sampling rate in Hz
t      = 0:1/samp:1; % time (1s of data)
t      = t(1:end-1); % remove last time point
N      = length(t); % store the number of time points
x = cos(2*pi*freq1*t) + .5*cos(2*pi*freq2*t); % the signal equation
figure
plot(t,x,'.-');
F = 50;      % sub-sampling frequency (Hz)
T = 1/F;    % sub-sampling period (sec)
n = N*T;    % sub-sampling period (in samples)
z = x(1:n:end); % sub-sampled signal
y = fft(z); % fft of sub-sampled signal
subN = length(z); % length of sub-sampled signal
f = linspace(0,subN-1,subN); % vector of frequencies for plotting
figure
plot(f(1:subN/2),abs(y(1:subN/2))) % plot powerspectrum as before
```

Şimdi alt örnekleme sıklığını değiştirin ve fourier dönüşümüne ne olduğunu görün. 10Hz altına düştüğünüzde, powerspectrum'da zirveyi 5Hz'de kaybettiğinizi görebilmelisiniz. Sinyali fft'den yeniden yapılandırmak ve neredeyse sürekli sürümle karşılaştırmak için ifft işlevini kullanmaya çalışın. F=10 alın.

Sıfır dolgu:

Sıfır dolguyu anlamak için 2D FFT ve Apricot resmini kullanacağız. İlk önce görüntüyü yükleyin ve diğer pikselleri kaldırarak alt örnekleyin:

```
clear all
close all
im=imread('Apricot.png');
x=im(1:2:end,1:2:end,1);
figure
imshow(x)
y=fft2(x);
clim=quantile(abs(y(:)), [.01 .99]);
figure, imagesc(fftshift(abs(y)),clim);colormap gray
```

Fourier dönüşümünde tekrarlar süresi örnekleme periyodu ile ters orantılıdır. Sıfır doldurma, sinyali yeniden oluşturmadan önce FFT'nin her iki tarafına sıfırların eklenmesi anlamına gelir. Matlab FFT, sinyalin periyodik olduğunu varsaydığından, sıfırlama sinyali, daha yüksek tekrarlar periyodu ile sinyali tekrarlamak gibidir, bu da gerçek alanda daha yüksek örnekleme oranı anlamına gelir. Bu nedenle sıfır dolgusu, bir görüntüyü frekans alanının bize izin verdiği kadar yüksek çözünürlükte yeniden yapılandırılmamızı sağlar ve enterpolasyon kavramıyla yakından ilişkilidir.

```
y = fftshift(y); % we need this before zero padding
z = ifft2(y, size(im,1), size(im,2)); % zero padded ifft
clim=quantile(abs(z(:)), [.01 .99]);
figure, imshow(abs(z), clim); colormap gray
```

Orijinal görüntüyle karşılaştırıldığında, enterpolasyonun neden olduğu salınımlara dikkat edilmelidir.

Filtering:

Burada evrişim ve Fourier dönüşümünü kullanarak basit bir doğrusal filtre uygulayacağız. Gerçek uzayda evrişimin Fourier uzayda çarpma ile aynı olduğunu ve tam tersini hatırlayın. Ancak evrişimin çoğaltılmasından daha zordur, bu nedenle genellikle sinyalin ve filtrenin Fourier dönüşümünü yapmak, sonra iki FFT'yi birlikte çoğaltmak ve sonra ters bir FFT yapmak iyi bir fikirdir. Görüntünün ve filtrenin Fourier dönüşümünü hesaplayalım:

```
siz = [3 3]; % 3x3 box. You can play with changing this
box = ones(siz)/9;
x=im(:,:,1);
y=fft2(x);
f=fft2(box,size(x,1),size(x,2)); % use zeropadding
z=ifft2(y.*f); % inverse FFT of the product
figure,
clim=quantile(abs(z(:)), [.01 .99]);
imagesc(abs(z),clim);colormap gray
```

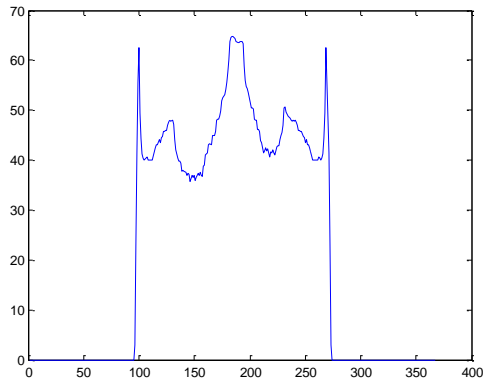
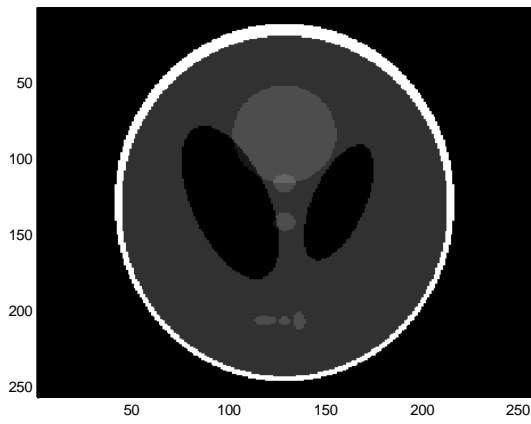
Gauss filtresi ortalama filtreye benzer, ancak kutudaki değerlerin sabit olması yerine, kutunun ortasında ve verilen standart sapmayla ortalanmış bir Gauss fonksiyonunu takip etmeleri dışında. Şimdi başka bir filtre tipi tasarlayalım: bir kenar algılama filtresi. Bu tür en basit filtre, ayrı ayrı terimlerle, bitişik pikseller arasındaki fark olan bir türev yapmaktır.

```
box = [-1 1]; % along x-dimension
box = [-1;1]; % along y-dimension
box = [-1 0 1]; % a slightly better filter along x
box = [-1;0;1]; % a slightly better filter along y
x=im(:,:,1);
y=fft2(x);
box = [-1 0 1]; % try with different ones if you have time
f=fft2(box,size(x,1),size(x,2)); % use zeropadding
z=ifft2(y.*f); % inverse FFT of the product
figure,
clim=quantile(abs(z(:)), [.01 .99]);
imagesc(abs(z),clim);colormap gray
```

Image reconstruction - Computed Tomography

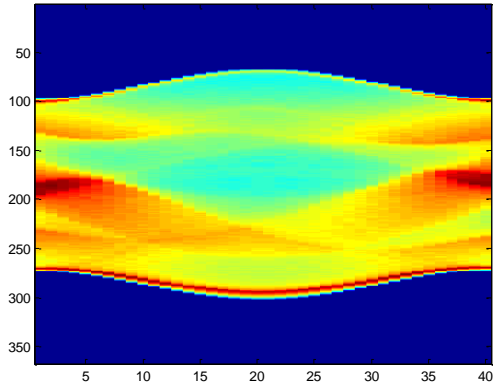
İlk olarak, bir nesneye ihtiyacımız var. İki boyutta kalacağız ve tıbbi görüntüleme en ünlü görüntüyü kullanacağız: Shepp-Logan fantomu. Bu resim o kadar popüler ki Matlab'ın görüntüleme araç kutusuna dahil edildi, bu yüzden onu oluşturalım:

```
clear all
close all
im=phantom;
figure, imagesc(im), colormap gray
R = radon(im,0);
figure, plot(R);
```



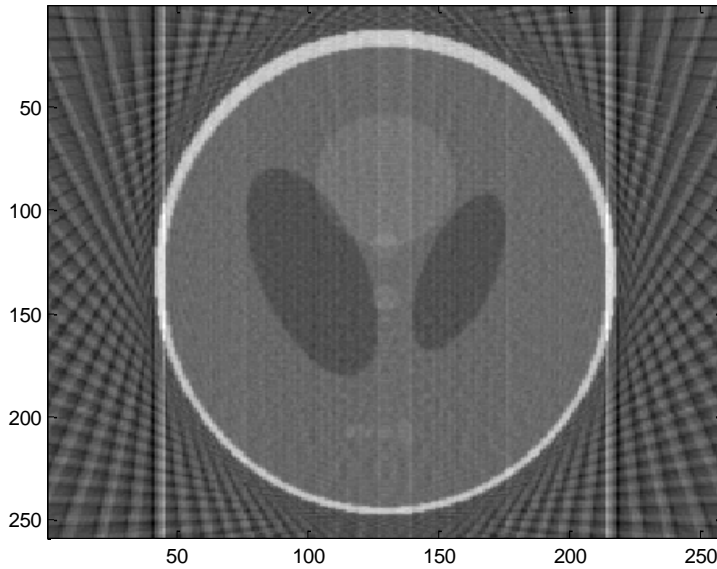
Şimdi daha fazla açı boyunca Radon dönüşümüne bakalım ve sonuçları çizelim:

```
R = radon(im,linspace(0,180,40)); % 40 angles from 0 to 180 degrees
figure, imagesc(R);
```



Şimdi ters bir Radon dönüşümü hesaplayalım ve nesnenin neye benzediğini görelim:

```
x = iradon(R,linspace(0,180,40));
figure,imagesc(x),colormap gray
```



Ters Radon transformına geri projeksiyon da denir. Nesneyi yeniden yapılandırmaya çalıştığınızda bu arka izdüşümün geride "izler" bıraktığını açıkça görebilirsiniz. Kaç tanesinin yeterli olduğunu anlamak için farklı projeksiyonlarla yukarıdakileri deneyin. Ayrıca, nesnedeki küçük ayrıntıların düzgün bir şekilde kurtarılması için daha fazla açı gerektirdiğine dikkat edin.

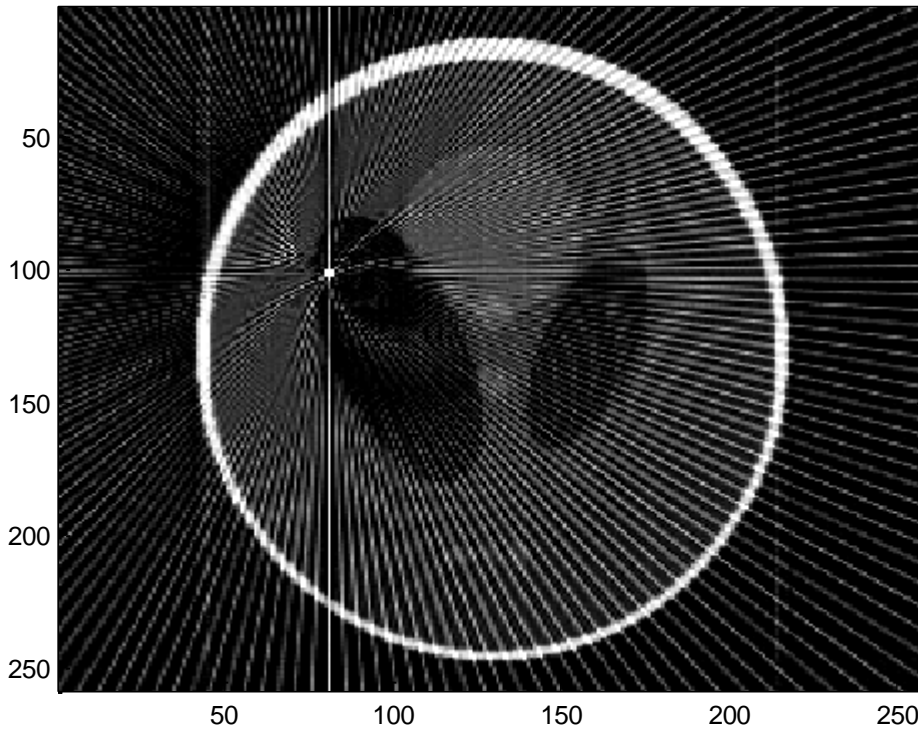
Bilgisayarlı Tomografide ölçüm aslında nesnenin X-ışını huzmesi boyunca tam olarak integrali değil, ışınlar boyunca "emilim" in integralidir. Ama bunu görmezden geleceğiz ve nesnenin

kütlesinin integralini ölçüyormuşuz gibi davranacağız. Sonra bir "metal" arayacağız. Hastanın vücudunun içinde bir implant olduğunda olur. Orijinal fantom görüntüsünde bir ani yükselterek bunu modelleyebiliriz:

Şimdi radon dönüşümünü (ölçüm işlemi simüle etmek için) ve sonra radon dönüşümünü (nesnenin ölçümlerden yeniden yapılandırılması) çalıştırın. Simüle edilmiş metal implantın neden olduğu bazı çarpıcı etkileri görebilmelisiniz.

```
im(100,80)=100;  
R = radon(im,linspace(0,180,100));  
x = iradon(R,linspace(0,180,100));  
figure,imagesc(x,[0 1]);colormap gray
```

Radon dönüşümü, projeksiyon-dilim teoremi aracılığıyla Fourier dönüşümü ile ilgilidir. Bu teorem, bir nesneniz varsa (örneğin 2B'de), nesneyi bir çizgiye yansıtmanın, nesnenin 2B Fourier dönüşümünden bir dilim almakla aynı olduğunu belirtir.



Not: metal olduğu bir önceki resimdeki sdairenin dışındaki saçılmalardan anlaşılmaktadır.

4. Laplace Transformation

Laplace dönüşümü diferansiyel denklemlerin çözümünde ortaya çıkmaktadır:

- Bir diferansiyel denklemi cebirsel bir denkleme dönüştürür. Böylece, daha basit cebirsel denklemleri çözerek ve sonra dönüştürmeyi tersine çevirerek, diferansiyel denkleme bir çözüm üretebiliriz.
- Bir rastgele değişkenin moment üretme fonksiyonu, yoğunluğunun Laplace dönüşümü ile ilgilidir (-s ile değiştirerek) ve toplamın yoğunluğunu dikkate alırken yararlı olabilir çünkü bir konvolüsyonun Laplace dönüşümü onların Laplace dönüşümlerinin ürünüdür.

Fourier dönüşümü, sadece sinüs ve kosinüs temel işlevleri kullanılarak sentezlenebilen sinyalleri analiz etmek için yeterlidir. Ancak, sinyalin üstel bileşenleri olduğunu bulduğumuzda, örneğin, zaman içinde üssel olarak değişkenlik gösteren sinüs dalgası için, bu, Fourier'ın verdiği bize sadece yarım bilgi. Böyle durumlarda kaybolur. Bu tür durumlarda Laplace dönüşümü zorunludur.

Laplace, filtre tasarımında ve analog devrelerin analizinde yaygın olarak kullanılır.

Laplace, hangi üssel koşulların kullanılması gerektiğine karar verebilir, böylece sistemimiz birleşir ve aynı zamanda sabit kalır.

Aniden başlayan veya biten herhangi bir şeyi analiz etmek isterseniz, Laplace dönüşümünden büyük ölçüde faydalanırsınız. Bu geçişlere sistem cevabı gibi gerçek uygulamalarda çok karşılaşılar. Aksine, Fourier dönüşümü, bir Laplace dönüşümünün problemi büyük ölçüde basitleştirdiği asimetrik fenomenlerle uğraşırken çok verimsizdir. Bu, Laplace'in çok önemli bir analitik aracı dönüştürmesini sağlar.

Laplace transformasyonu doğrusal sistemleri tanımlamak için yaygın olarak kullanılmaktadır. Türevleri içeren dağınık denklemleri cebirsel denklemlere dönüştürebilir. Eğer sistem bir giriş ve bir çıkışa sahipse, çıkışın girişin bir fonksiyonu olarak tanımlandığı durumlarda, onu s etki alanına (Laplace kullanarak) dönüştürebilir ve sistemi bir transfer fonksiyonu ile tanımlayabilirsiniz. $X(s)$ 'in girdi olduğunu, $Y(s)$ 'nin çıkış olduğunu ve $H(s)$ 'nin transfer fonksiyonu olarak bilinen şey olduğunu varsayalım. Sistem çıkışı $Y(s) = H(s) * X(s)$ olarak tanımlanabilir. Transfer fonksiyonu $Y(s) / X(s)$ ile eşit olacaktır. Bu önemli görünmeyebilir, ancak bir sistemin transfer fonksiyonu bulunduğu anda, bize söylenen sistemin çok fazla bir kısmını söyleyebilir. Transfer fonksiyonunun pay ve paydası faktörize edilirse, bir kutup-sıfır tanımı yapılabilir. Sisteminiz yüksek frekansları söndürmek için yapılmış bir elektrik devresi ise, düşük geçiş filtresi, sadece kutup-sıfır çizimi filtrenin belirli frekanslara nasıl tepki

verdiğini size söyleyebilir. Bu, laplace dönüşümünün sadece küçük bir uygulamasıdır, ancak birçok mühendislik dalında kullanışlı olabilen transfer fonksiyonları ile lineer sistemlerin tanımlanmasını sağlar.

- Sistemlerin, sadece bir frekans veya zaman alanı verileri yerine, sinusoidler ve üsteller gibi doğal bileşenler açısından analiz edilmesine yardımcı olur.
- Yoğun aktarım işlevini, kutuplar ve sıfırlar cinsinden tanımlanabildiği ve sezgisel olarak sistemin kararlılığını, aşılmasını veya gürültüsünü tahmin etmede yardımcı olan uygun bir alana (S-etki alanı) çevirir.
- Fourier Dönüşümü, şifreli bir dönüşümü çok daha kolay bir çarpıma dönüştürürken, bir Laplace Dönüşümü, S-etki alanında basit bir polinom cebiri ile yoğun bir diferansiyel denklemin çözülmesine yardımcı olur.
- Yinelemeli filtrelerin kararlılığını analiz etmek için DSP'de temel olan Z-Dönüşümü, Laplace Dönüşümü'nün yakın akrabasıdır.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t) dt \quad s > 0$$

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt$$

as the Laplace transform of $f(t)$, and the inverse Laplace transform is

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s)e^{st} ds$$

Theorem: The conditions for the existence of Laplace transform of $f(t)$ are

(1) $f(t)$ is piecewise continuous in every finite interval.

(2) $f(t)$ is an exponential order function.

Laplace Dönüşümleri

- 1) Dinamik ve kontrolde standart gösterimi (kısa gösterim) sağlar.
- 2) Matematik cebirsel işlemlere dönüştürür.
- 3) Blok diyagram analizi için avantajlıdır.

Laplace dönüşümleri proses kontrolünde kullanılabilir:

- 1) Diferansiyel denklemlerin çözümü (doğrusal)
- 2) Doğrusal kontrol sistemlerinin analizi (frekans cevabı)
- 3) Farklı girdiler için geçici yanıtın tahmini

4.1. Laplace Transform of Basic Functions

Original function	Transformed function
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$
e^{at}	$\frac{1}{s-a}$

Original function	Transformed function
$\sin at$	$\frac{a}{s^2 + a^2}$
$\cos at$	$\frac{s}{s^2 + a^2}$
$\sinh at$	$\frac{a}{s^2 - a^2}$
$\cosh at$	$\frac{s}{s^2 - a^2}$

$$1. \mathcal{L} [1] = \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}$$

$$2. \mathcal{L} [t^a] = \int_0^{\infty} t^a e^{-st} dt = \int_0^{\infty} \left(\frac{u}{s}\right)^a e^{-u} \frac{du}{s} = \frac{1}{s^{a+1}} \int_0^{\infty} u^a e^{-u} du = \frac{\Gamma(a+1)}{s^{a+1}}$$

$$3. \mathcal{L} [e^{at}] = \int_0^{\infty} e^{at} e^{-st} dt = \frac{e^{-(s-a)t}}{-(s-a)} \Big|_0^{\infty} = \frac{1}{s-a}$$

$$4. \mathcal{L} [e^{iat}] = \frac{1}{s-ia} \Rightarrow \mathcal{L} [\cos at + i \sin at] = \frac{s}{s^2 + a^2} + i \frac{a}{s^2 + a^2}$$

$$\therefore \mathcal{L} [\cos at] = \frac{s}{s^2 + a^2}, \text{ and } \mathcal{L} [\sin at] = \frac{a}{s^2 + a^2}$$

$$5. \mathcal{L} [\sinh at] = \mathcal{L} \left[\frac{e^{at} - e^{-at}}{2} \right] = \frac{1}{2} \left(\frac{1}{s-a} - \frac{1}{s+a} \right) = \frac{a}{s^2 - a^2}$$

$$\mathcal{L} [\cosh at] = \mathcal{L} \left[\frac{e^{at} + e^{-at}}{2} \right] = \frac{1}{2} \left(\frac{1}{s-a} + \frac{1}{s+a} \right) = \frac{s}{s^2 - a^2}$$

$\Gamma(x) = \int_0^{\infty} e^{-u} u^{x-1} du$ is called **Gamma function**.

The properties of Gamma function

$$\Gamma(a+1) = a\Gamma(a).$$

$$\Gamma(1) = 1.$$

$\Gamma(n+1) = n!$, n is a natural number.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

$$\Gamma(a+1) = \int_0^{\infty} e^{-u} u^{(a+1)-1} du = \int_0^{\infty} e^{-u} u^a du = -(e^{-u} u^a \Big|_0^{\infty} - a \int_0^{\infty} e^{-u} u^{a-1} du) = a \int_0^{\infty} e^{-u} u^{a-1} du = a\Gamma(a)$$

$$\Gamma(1) = \int_0^{\infty} e^{-u} u^{1-1} du = \int_0^{\infty} e^{-u} du = -e^{-u} \Big|_0^{\infty} = 1$$

When n is a natural number, then

$$\begin{aligned} \Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2) \\ &= \dots = n(n-1)(n-2)\dots \times 2 \times 1 \times \Gamma(1) = n! \end{aligned}$$

$$\begin{aligned} \Gamma\left(\frac{1}{2}\right) &= \int_0^{\infty} e^{-u} u^{\frac{1}{2}-1} du = \int_0^{\infty} e^{-u} u^{-\frac{1}{2}} du = 2 \int_0^{\infty} e^{-u} d\sqrt{u} = 2 \int_0^{\infty} e^{-x^2} dx = 2 \sqrt{\int_0^{\infty} e^{-x^2} dx \cdot \int_0^{\infty} e^{-y^2} dy} \\ &= 2 \sqrt{\int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy} = 2 \sqrt{\int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} r dr d\theta} = 2 \sqrt{\frac{1}{2} \int_0^{\frac{\pi}{2}} (-e^{-r^2})_0^{\infty} d\theta} = 2 \sqrt{\frac{1}{2} \int_0^{\frac{\pi}{2}} 1 d\theta} \\ &= 2 \sqrt{\frac{1}{2} \cdot \frac{\pi}{2}} = \sqrt{\pi} \end{aligned}$$

4.2. Laplace Transform of Special Functions

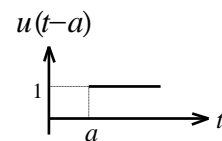
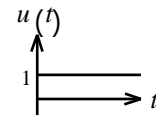
1. Unit step function (Heaviside function)

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\begin{aligned} \mathcal{L} [u(t)] &= \int_0^{\infty} u(t)e^{-st} dt \\ &= \int_0^{\infty} e^{-st} dt = \frac{1}{s} \end{aligned}$$

$$\Rightarrow u(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$$

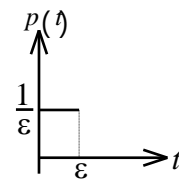
$$\mathcal{L} [u(t-a)] = \frac{e^{-as}}{s}$$



2. Unit impulse function (Dirac delta function)

$$(1) \text{ square wave function } p(t) = \begin{cases} \frac{1}{\varepsilon} & 0 \leq t \leq \varepsilon \\ 0 & t > \varepsilon \end{cases}$$

$$\begin{aligned} \mathcal{L} [p(t)] &= \int_0^{\infty} p(t)e^{-st} dt = \int_0^{\varepsilon} \frac{1}{\varepsilon} [u(t) - u(t-\varepsilon)]e^{-st} dt \\ &= \frac{1}{\varepsilon} \left(\frac{1}{s} - \frac{e^{-\varepsilon s}}{s} \right) = \frac{1 - e^{-\varepsilon s}}{s\varepsilon} \end{aligned}$$



(2) unit impulse function $\delta(t) = \lim_{\varepsilon \rightarrow 0} p(t)$ (also called singular function, Dirac delta function)

$$\mathcal{L} [\delta(t)] = \mathcal{L} [\lim_{\varepsilon \rightarrow 0} p(t)] = \lim_{\varepsilon \rightarrow 0} \{ \mathcal{L} [p(t)] \} = \lim_{\varepsilon \rightarrow 0} \frac{1 - e^{-\varepsilon s}}{s\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{se^{-\varepsilon s}}{s} = 1$$

Properties:

$$(i) \int_0^{\infty} \delta(t) dt = u(t)$$

$$(ii) \int_0^{\infty} g(t)\delta(t-a) dt = g(a)$$

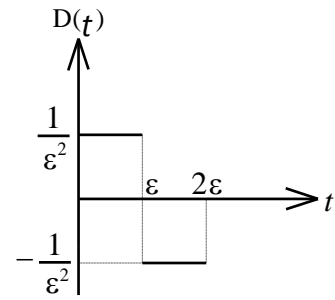
$$(iii) \int_0^{\infty} g(t)\delta(t) dt = g(0)$$

$$\mathcal{L} [\delta(t-a)] = e^{-as}$$

3. Unit double function

$$D(t) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} [u(t) - 2u(t - \varepsilon) + u(t - 2\varepsilon)]$$

$$\begin{aligned} \mathcal{L} [D(t)] &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left(\frac{1}{s} - \frac{2e^{-\varepsilon s}}{s} + \frac{e^{-2\varepsilon s}}{s} \right) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1 - 2e^{-\varepsilon s} + e^{-2\varepsilon s}}{\varepsilon^2 s} = \lim_{\varepsilon \rightarrow 0} \frac{2se^{-\varepsilon s} - 2se^{-2\varepsilon s}}{2\varepsilon s} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{-2s^2 e^{-\varepsilon s} + 4s^2 e^{-2\varepsilon s}}{2s} = s \end{aligned}$$



Question:

Let $x(t) = e^{-3t} [H(t) * \delta(t)]$, where $H(t)$ is unit step function, and $\delta(t)$ is Dirac delta function,

and $*$ represents convolution, find (a) $\frac{d^2 x}{dt^2}$ (b) $\mathcal{L} [x(t)]$

$$\text{Solution : (a) } H(t) * \delta(t) = \int_0^t H(\tau) \delta(t - \tau) d\tau = H(t)$$

$$\therefore x(t) = e^{-3t} H(t)$$

$$\frac{dx}{dt} = e^{-3t} [-3H(t) + \delta(t)]$$

$$\frac{d^2 x}{dt^2} = e^{-3t} \{-3[-3H(t) + \delta(t)] - 3\delta(t) + \delta'(t)\}$$

$$= e^{-3t} [9H(t) - 6\delta(t) + \delta'(t)]$$

$$(b) \mathcal{L} [x(t)] = \mathcal{L} [e^{-3t} H(t)] = \frac{1}{s + 3}$$

4.3. Properties of Laplace Transform

Property	Original Function	Transformed Function
Linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
Shifting	$f(t-a)u(t-a)$	$e^{-as}F(s)$
	$e^{at}f(t)$	$F(s-a)$
Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Differentiation	$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \dots - f^{(n-1)}(0)$
	$(-t)^n f(t)$	$\frac{d^n F(s)}{ds^n}$
Integration	$\int_0^t f(\tau) d\tau$	$\frac{1}{s} F(s)$
	$\frac{1}{t} f(t)$	$\int_s^\infty F(s) ds$
Convolution	$\int_0^t f(\tau)g(t-\tau) d\tau$	$F(s)G(s)$
Periodic Function	$f(t) = f(t+T)$	$\frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st} dt$
Initial Value Theorem	$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$	
	$\lim_{t \rightarrow 0} \frac{f(t)}{g(t)} = \lim_{s \rightarrow \infty} \frac{F(s)}{G(s)}$	
Final Value Theorem	$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$	
	$\lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} = \lim_{s \rightarrow 0} \frac{F(s)}{G(s)}$	

1. Linearity

$$\mathcal{L} [af(t) + bg(t)] = \int_0^{\infty} [af(t) + bg(t)]e^{-st} dt = a \int_0^{\infty} f(t)e^{-st} dt + b \int_0^{\infty} g(t)e^{-st} dt = aF(s) + bG(s)$$

Find the Laplace transform of $\cos^2 t$.

$$\text{Solution : } \mathcal{L} [\cos^2 t] = \mathcal{L} \left[\frac{1 + \cos 2t}{2} \right] = \frac{1}{2} \left(\frac{1}{s} + \frac{s}{s^2 + 2^2} \right) = \frac{s^2 + 2}{s(s^2 + 4)}$$

2. Shifting

$$(a) \mathcal{L} [f(t-a)u(t-a)] = \int_0^{\infty} f(t-a)u(t-a)e^{-st} dt = \int_a^{\infty} f(t-a)e^{-st} dt$$

Let $\tau = t - a$, then

$$\mathcal{L} [f(t-a)u(t-a)] = \int_0^{\infty} f(\tau)e^{-s(\tau+a)} d\tau = e^{-sa} \int_0^{\infty} f(\tau)e^{-s\tau} d\tau = e^{-sa} F(s)$$

$$(b) F(s-a) = \int_0^{\infty} f(t)e^{-(s-a)t} dt = \int_0^{\infty} [e^{at} f(t)]e^{-st} dt = \mathcal{L} [e^{at} f(t)]$$

What is the Laplace transform of the function: $f(t) = \begin{cases} 0, & t < 4 \\ 2t^3, & t \geq 4 \end{cases}$.

Solution: $f(t) = 2t^3 u(t-4)$

$$\begin{aligned} \mathcal{L} [f(t)] &= \mathcal{L} \{ 2[(t-4)^3 + 12(t-4)^2 + 48(t-4) + 64] u(t-4) \} \\ &= 2e^{-4s} \left(\frac{3!}{s^4} + 12 \times \frac{2!}{s^3} + 48 \times \frac{1}{s^2} + \frac{64}{s} \right) = 4e^{-4s} \left(\frac{3}{s^4} + \frac{12}{s^3} + \frac{24}{s^2} + \frac{32}{s} \right) \end{aligned}$$

3. Scaling

$$\mathcal{L} [f(at)] = \int_0^{\infty} f(at)e^{-st} dt$$

Let $\tau = at$, then

$$\mathcal{L} [f(at)] = \int_0^{\infty} f(\tau)e^{-\frac{s}{a}\tau} \frac{\tau}{a} d\tau = \frac{1}{a} \int_0^{\infty} f(\tau)e^{-\frac{s}{a}\tau} d\tau = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Find the Laplace transform of $\cos 2t$.

$$\text{Solution : } \mathcal{L} [\cos t] = \frac{s}{s^2 + 1}$$

$$\therefore \mathcal{L} [\cos 2t] = \frac{1}{2} \frac{\frac{s}{2}}{\left(\frac{s}{2}\right)^2 + 1} = \frac{s}{s^2 + 4}$$

4. Derivative

(a) Derivative of original function

$$\mathcal{L} [f'(t)] = \int_0^{\infty} f'(t)e^{-st} dt = f(t)e^{-st} \Big|_0^{\infty} - (-s) \int_0^{\infty} f(t)e^{-st} dt$$

(1) If $f(t)$ is continuous, equation reduces to

$$\mathcal{L} [f'(t)] = -f(0) + sF(s) = sF(s) - f(0)$$

(2) If $f(t)$ is not continuous at $t = a$, equation reduces to

$$\begin{aligned} \mathcal{L} [f'(t)] &= f(t)e^{-st} \Big|_0^{a^-} + f(t)e^{-st} \Big|_{a^+}^{\infty} + sF(s) = [f(a^-)e^{-sa} - f(0)] + [0 - f(a^+)e^{-sa}] + sF(s) \\ &= sF(s) - f(0) - e^{-sa} [f(a^+) - f(a^-)] \end{aligned}$$

(3) Similarly, if $f(t)$ is not continuous at $t = a_1, a_2, \dots, a_n$, equation reduces to

$$\mathcal{L} [f'(t)] = sF(s) - f(0) - \sum_{i=1}^n e^{-sa_i} [f(a_i^+) - f(a_i^-)]$$

[Deduction] If $f(t), f'(t), f''(t), \dots, f^{(n-1)}(t)$ are continuous, and $f^{(n)}(t)$ is piecewise continuous, and all of them are exponential order functions, then

$$\mathcal{L} [f^{(n)}(t)] = s^n F(s) - \sum_{i=1}^n s^{n-i} f^{(i-1)}(0)$$

(b) Derivative of transformed function

$$\frac{dF(s)}{ds} = \frac{d}{ds} \int_0^{\infty} f(t)e^{-st} dt = \int_0^{\infty} \frac{\partial}{\partial s} [f(t)e^{-st}] dt = \int_0^{\infty} (-t)f(t)e^{-st} dt = \mathcal{L} [(-t)f(t)]$$

$$\text{[Deduction]} \quad \frac{d^n F(s)}{ds^n} = \mathcal{L} [(-t)^n f(t)]$$

Question:

Find the Laplace transform of te^t .

$$\text{Solution : } \mathcal{L} (e^t) = \frac{1}{s-1} \Rightarrow \mathcal{L} (te^t) = -\frac{d}{ds} \left(\frac{1}{s-1} \right) = \frac{1}{(s-1)^2}$$

Question:

$$f(t) = \begin{cases} t^2, & 0 \leq t \leq 1 \\ 0, & t > 1 \end{cases}, \text{ find } \mathcal{L} [f'(t)].$$

Solution : $f(t) = t^2[u(t) - u(t-1)]$

$$\begin{aligned} \mathcal{L} [f(t)] &= \mathcal{L} [t^2u(t)] - \mathcal{L} [t^2u(t-1)] = \frac{2!}{s^3} - \mathcal{L} \{[(t-1) + 1]^2u(t-1)\} \\ &= \frac{2}{s^3} - \mathcal{L} \{[(t-1)^2 + 2(t-1) + 1]u(t-1)\} \\ &= \frac{2}{s^3} - e^{-s} \left(\frac{2}{s^3} + 2 \frac{1}{s^2} + \frac{1}{s} \right) \end{aligned}$$

$$\begin{aligned} \mathcal{L} [f'(t)] &= sF(s) - f(0) - e^{-s}[f(1^+) - f(1^-)] \\ &= \left[\frac{2}{s^2} - e^{-s} \left(\frac{2}{s^2} + \frac{2}{s} + 1 \right) \right] - 0 - e^{-s}(0 - 1) = \frac{2}{s^2} - e^{-s} \left(\frac{2}{s^2} + \frac{2}{s} \right) \end{aligned}$$

5. Integration

(a) Integral of original function

$$\begin{aligned} \mathcal{L} \left[\int_0^t f(\tau) d\tau \right] &= \int_0^\infty \int_0^t f(\tau) d\tau e^{-st} dt \\ &= \frac{1}{-s} \left[e^{-st} \int_0^t f(\tau) d\tau \Big|_0^\infty - \int_0^\infty f(t) e^{-st} dt \right] = \frac{1}{s} F(s) \end{aligned}$$

$$\Rightarrow \mathcal{L} \left[\int_0^t \int_0^t \dots \int_0^t f(t) dt dt \dots dt \right] = \frac{1}{s^n} F(s)$$

(b) Integration of Laplace transform

$$\begin{aligned} \int_s^\infty F(s) ds &= \int_s^\infty \int_0^\infty f(t) e^{-st} dt ds = \int_0^\infty f(t) \int_s^\infty e^{-st} ds dt \\ &= \int_0^\infty f(t) \frac{e^{-st}}{-t} \Big|_s^\infty dt = \int_0^\infty \frac{f(t)}{t} e^{-st} dt = \mathcal{L} \left[\frac{f(t)}{t} \right] \end{aligned}$$

$$\Rightarrow \int_s^\infty \int_s^\infty \dots \int_s^\infty F(s) ds ds \dots ds = \mathcal{L} \left[\frac{1}{t^n} f(t) \right]$$

Question:

Find (a) $\mathcal{L} \left[\frac{1-e^{-t}}{t} \right]$ (b) $\mathcal{L} \left[\frac{1-e^{-t}}{t^2} \right]$.

Solution : (a) $\mathcal{L} [1 - e^{-t}] = \frac{1}{s} - \frac{1}{s+1}$

$$\begin{aligned} \mathcal{L} \left[\frac{1-e^{-t}}{t} \right] &= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1} \right) ds = \ln s - \ln(s+1) \Big|_s^\infty = \ln \frac{s}{s+1} \Big|_s^\infty \\ &= 0 - \ln \frac{s}{s+1} = \ln \frac{s+1}{s} \end{aligned}$$

$$\begin{aligned} (b) \mathcal{L} \left[\frac{1-e^{-t}}{t^2} \right] &= \int_s^\infty \ln \frac{s+1}{s} ds = s \ln \frac{s+1}{s} \Big|_s^\infty - \int_s^\infty s \left(\frac{1}{s+1} - \frac{1}{s} \right) ds \\ &= s \ln \frac{s+1}{s} \Big|_s^\infty + \int_s^\infty \frac{1}{s+1} ds = \left[s \ln \frac{s+1}{s} + \ln(s+1) \right]_s^\infty \\ &= [(s+1) \ln(s+1) - s \ln s]_s^\infty = s \ln s - (s+1) \ln(s+1) \end{aligned}$$

Question:

Find (a) $\int_0^\infty \frac{\sin kt e^{-st}}{t} dt$ (b) $\int_{-\infty}^\infty \frac{\sin x}{x} dx$.

Solution : (a) $\int_0^\infty \frac{\sin kte^{-st}}{t} dt = \mathcal{L} \left[\frac{\sin kt}{t} \right]$

$$\therefore \mathcal{L} [\sin kt] = \frac{k}{s^2 + k^2}$$

$$\mathcal{L} \left[\frac{\sin kt}{t} \right] = \int_s^\infty \frac{k}{s^2 + k^2} ds = \frac{1}{k} \int_s^\infty \frac{1}{\left(\frac{s}{k}\right)^2 + 1} ds$$

$$= \tan^{-1} \frac{s}{k} \Big|_s^\infty = \frac{\pi}{2} - \tan^{-1} \frac{s}{k}$$

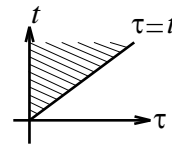
$$(b) \int_{-\infty}^\infty \frac{\sin x}{x} dx = 2 \int_0^\infty \frac{\sin x}{x} dx$$

$$= 2 \lim_{\substack{k \rightarrow 1 \\ s \rightarrow 0}} \int_0^\infty \frac{\sin kte^{-st}}{t} dt$$

$$= 2 \lim_{\substack{k \rightarrow 1 \\ s \rightarrow 0}} \left(\frac{\pi}{2} - \tan^{-1} \frac{s}{k} \right) = \pi$$

6. Convolution theorem

$$\begin{aligned} \mathcal{L} \left[\int_0^t f(\tau)g(t-\tau)d\tau \right] &= \int_0^\infty \int_0^t f(\tau)g(t-\tau)d\tau e^{-st} dt \\ &= \int_0^\infty \int_\tau^\infty f(\tau)g(t-\tau)e^{-st} dt d\tau = \int_0^\infty f(\tau) \int_\tau^\infty g(t-\tau)e^{-st} dt d\tau \end{aligned}$$



Let $u = t - \tau$, $du = dt$, then

$$\begin{aligned} \mathcal{L} \left[\int_0^t f(\tau)g(t-\tau)d\tau \right] &= \int_0^\infty f(\tau) \int_0^\infty g(u)e^{-s(u+\tau)} du d\tau \\ &= \int_0^\infty f(\tau)e^{-s\tau} d\tau \int_0^\infty g(u)e^{-su} du = F(s)G(s) \end{aligned}$$

Question:

Find the Laplace transform of $\int_0^t e^{t-\tau} \sin 2\tau d\tau$.

$$\text{Solution : } \because \mathcal{L} [e^t] = \frac{1}{s-1}, \mathcal{L} [\sin 2t] = \frac{2}{s^2 + 4}$$

$$\begin{aligned} \therefore \mathcal{L} \left[\int_0^t e^{t-\tau} \sin 2\tau d\tau \right] &= \mathcal{L} [e^t * \sin 2t] = \mathcal{L} [e^t] \cdot \mathcal{L} [\sin 2t] \\ &= \frac{1}{s-1} \cdot \frac{2}{s^2 + 4} = \frac{2}{(s-1)(s^2 + 4)} \end{aligned}$$

7. Periodic Function: $f(t+T) = f(t)$

$$\mathcal{L} [f(t)] = \int_0^\infty f(t)e^{-st} dt = \int_0^T f(t)e^{-st} dt + \int_T^{2T} f(t)e^{-st} dt + \dots$$

$$\text{and } \int_T^{2T} f(t)e^{-st} dt = \int_0^T f(u+T)e^{-s(u+T)} du = e^{-sT} \int_0^T f(u)e^{-su} du$$

Similarly,

$$\int_{2T}^{3T} f(t)e^{-st} dt = e^{-2sT} \int_0^T f(u)e^{-su} du$$

$$\begin{aligned} \therefore \mathcal{L} [f(t)] &= (1 + e^{-sT} + e^{-2sT} + \dots) \int_0^T f(t)e^{-st} dt \\ &= \frac{1}{1 - e^{-sT}} \int_0^T f(t)e^{-st} dt \end{aligned}$$

Question:

Find the Laplace transform of $f(t) = \frac{k}{p}t, 0 < t < p, f(t+p) = f(t)$.

$$\begin{aligned} \text{Solution : } \mathcal{L} [f(t)] &= \frac{1}{1 - e^{-ps}} \int_0^p \frac{k}{p} t e^{-st} dt \\ &= \frac{1}{1 - e^{-ps}} \frac{k}{p} \left[\frac{1}{-s} (te^{-st}) \Big|_0^p - \int_0^p e^{-st} dt \right] \\ &= \frac{-k}{ps(1 - e^{-ps})} \left(te^{-st} + \frac{1}{s} e^{-st} \right) \Big|_0^p \\ &= \frac{-k}{ps(1 - e^{-ps})} \left(pe^{-sp} + \frac{e^{-sp}}{s} - \frac{1}{s} \right) \end{aligned}$$

8. Initial Value Theorem:

$$\because \mathcal{L} [f'(t)] = sF(s) - f(0) \Rightarrow \lim_{s \rightarrow \infty} \int_0^{\infty} f'(t) e^{-st} dt = \lim_{s \rightarrow \infty} sF(s) - f(0) \Rightarrow 0 = \lim_{s \rightarrow \infty} sF(s) - f(0)$$

$$\text{we get initial value theorem } \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

$$\text{Deduce general initial value theorem: } \lim_{t \rightarrow 0} \frac{f(t)}{g(t)} = \lim_{s \rightarrow \infty} \frac{F(s)}{G(s)}$$

9. Final Value Theorem:

$$\mathcal{L} [f'(t)] = sF(s) - f(0) \Rightarrow \lim_{s \rightarrow 0} \int_0^{\infty} f'(t) e^{-st} dt = \lim_{s \rightarrow 0} sF(s) - f(0) \Rightarrow$$

$$\lim_{t \rightarrow \infty} f(t) - f(0) = \lim_{s \rightarrow 0} sF(s) - f(0) \Rightarrow \text{final value theorem: } \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

$$\text{General final value theorem: } \lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} = \lim_{s \rightarrow 0} \frac{F(s)}{G(s)}$$

Question:

Find $\mathcal{L} \left[\int_0^t \frac{\sin x}{x} dx \right]$.

Solution : Let $f(t) = \int_0^t \frac{\sin x}{x} dx \Rightarrow f'(t) = \frac{\sin t}{t}, f(0) = 0$

$$\mathcal{L} [tf'(t)] = \mathcal{L} [\sin t] = \frac{1}{s^2 + 1}$$

$$-\frac{d}{ds} \mathcal{L} [f'(t)] = \frac{1}{s^2 + 1}$$

$$-\frac{d}{ds} [sF(s) - f(0)] = \frac{1}{s^2 + 1} \Rightarrow \frac{d}{ds} [sF(s)] = -\frac{1}{s^2 + 1}$$

$$sF(s) = -\tan^{-1}s + C$$

From the initial value theorem, we get

$$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

$$0 = -\frac{\pi}{2} + C \quad \therefore C = \frac{\pi}{2}$$

$$sF(s) = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1} \frac{1}{s}$$

$$F(s) = \frac{1}{s} \tan^{-1} \frac{1}{s}$$

4.4. Inverse Laplace Transform

1. Linearity

$$(a) \mathcal{L}^{-1}\left[\frac{2s+1}{s^2+4}\right] \quad (b) \mathcal{L}^{-1}\left[\frac{4(s+1)}{s^2-16}\right].$$

$$\text{Solution : (a) } \mathcal{L}^{-1}\left[\frac{2s+1}{s^2+4}\right] = \mathcal{L}^{-1}\left[2\frac{s}{s^2+2^2} + \frac{1}{2}\frac{2}{s^2+2^2}\right] = 2\cos 2t + \frac{1}{2}\sin 2t$$

$$(b) \mathcal{L}^{-1}\left[\frac{4(s+1)}{s^2-16}\right] = \mathcal{L}^{-1}\left[4\frac{s}{s^2-4^2} + \frac{4}{s^2-4^2}\right] = 4\cosh 4t + \sinh 4t$$

2. Shifting

$$(a) \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+2s+2}\right] \quad (b) \mathcal{L}^{-1}\left[\frac{2s+3}{s^2+3s+2}\right].$$

$$\text{Solution : (a) } \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+2s+2}\right] = \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{(s+1)^2+1}\right]$$

$$\therefore \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t} \sin t$$

$$\text{and } \mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$$

$$\therefore \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{(s+1)^2+1}\right] = e^{-(t-\pi)} \sin(t-\pi)u(t-\pi) = -e^{-(t-\pi)} \sin tu(t-\pi)$$

$$(b) \mathcal{L}^{-1}\left[\frac{2s+3}{s^2+3s+2}\right] = \mathcal{L}^{-1}\left[\frac{2(s+\frac{3}{2})}{(s+\frac{3}{2})^2 - (\frac{1}{2})^2}\right] = 2e^{-\frac{3}{2}t} \cosh \frac{t}{2}$$

3. Scaling

$$\mathcal{L}^{-1}\left[\frac{4s}{16s^2-4}\right].$$

$$\text{Solution : } \mathcal{L}^{-1}\left[\frac{4s}{16s^2-4}\right] = \mathcal{L}^{-1}\left[\frac{4s}{(4s)^2-2^2}\right] = \frac{1}{4}\cosh 2 \cdot \frac{1}{4}t = \frac{1}{4}\cosh \frac{t}{2}$$

4. Derivative

$$(a) \mathcal{L}^{-1}\left[\frac{1}{(s^2 + \omega^2)^2}\right] \quad (b) \mathcal{L}^{-1}\left[\ln \frac{s+a}{s+b}\right].$$

$$\text{solution : } (a) \mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} \Rightarrow \mathcal{L}[t \sin \omega t] = -\frac{d}{ds}\left(\frac{\omega}{s^2 + \omega^2}\right) = \frac{2\omega s}{(s^2 + \omega^2)^2}$$

$$\text{Let } F(t) = t \sin \omega t \Rightarrow \mathcal{L}[F'(t)] = s \cdot \frac{2\omega s}{(s^2 + \omega^2)^2} - F(0)$$

$$\mathcal{L}[F'(t)] = 2\omega \frac{s^2}{(s^2 + \omega^2)^2} = 2\omega \left[\frac{(s^2 + \omega^2) - \omega^2}{(s^2 + \omega^2)^2} \right] = 2\omega \left[\frac{1}{s^2 + \omega^2} - \frac{\omega^2}{(s^2 + \omega^2)^2} \right]$$

$$= 2\mathcal{L}[\sin \omega t] - \frac{2\omega^3}{(s^2 + \omega^2)^2}$$

$$\frac{1}{(s^2 + \omega^2)^2} = \frac{1}{2\omega^3} \cdot \mathcal{L}[2\sin \omega t - F'(t)]$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2 + \omega^2)^2}\right] = \frac{1}{2\omega^3} \cdot [2\sin \omega t - F'(t)] = \frac{1}{2\omega^3} (\sin \omega t - \omega t \cos \omega t)$$

$$(b) \text{ Let } \mathcal{L}[f(t)] = \ln \frac{s+a}{s+b} = \ln(s+a) - \ln(s+b)$$

$$\mathcal{L}[tf(t)] = -\frac{d}{ds} [\ln(s+a) - \ln(s+b)] = \frac{1}{s+b} - \frac{1}{s+a} = \mathcal{L}[e^{-bt} - e^{-at}]$$

$$\therefore f(t) = \frac{e^{-bt} - e^{-at}}{t}$$

5. Integration

$$(a) \mathcal{L}^{-1}\left[\frac{1}{s^2} \left(\frac{s-1}{s+1}\right)\right] \quad (b) \mathcal{L}^{-1}\left[\ln \frac{s+a}{s+b}\right].$$

$$\text{Solution : } (a) \mathcal{L}^{-1}\left[\frac{1}{s^2} \left(\frac{s-1}{s+1}\right)\right] = \mathcal{L}^{-1}\left[\frac{1}{s(s+1)} - \frac{1}{s^2(s+1)}\right] = \int_0^t e^{-t} dt - \int_0^t \int_0^t e^{-t} dt dt$$

$$= -(e^{-t} - 1) + \int_0^t (e^{-t} - 1) dt = -(e^{-t} - 1) - (e^{-t} - 1) - t = 2 - 2e^{-t} - t$$

$$(b) \mathcal{L}[e^{-bt} - e^{-at}] = \frac{1}{s+b} - \frac{1}{s+a}$$

$$\mathcal{L}\left[\frac{e^{-bt} - e^{-at}}{t}\right] = \int_s^\infty \left(\frac{1}{s+b} - \frac{1}{s+a}\right) ds = \ln \frac{s+b}{s+a} \Big|_s^\infty = \ln \frac{s+a}{s+b}$$

$$\therefore \mathcal{L}^{-1}\left[\ln \frac{s+a}{s+b}\right] = \frac{e^{-bt} - e^{-at}}{t}$$

6. Convolution

$$(a) \mathcal{L}^{-1}\left[\frac{1}{(s^2 + \omega^2)^2}\right] \quad (b) \mathcal{L}^{-1}\left[\frac{s}{(s^2 + \omega^2)^2}\right].$$

$$\text{Solution : (a) } \mathcal{L}^{-1}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} \Rightarrow \mathcal{L}^{-1}\left[\frac{1}{\omega} \sin \omega t\right] = \frac{1}{s^2 + \omega^2}$$

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{(s^2 + \omega^2)^2}\right] &= \frac{1}{\omega^2} \int_0^t \sin \omega \tau \sin \omega(t - \tau) d\tau \\ &= \frac{1}{\omega^2} \int_0^t \frac{1}{2} [\cos(\omega\tau - \omega t + \omega\tau) - \cos(\omega\tau + \omega t - \omega\tau)] d\tau \\ &= \frac{1}{2\omega^2} \int_0^t [\cos(2\omega\tau - \omega t) - \cos \omega t] d\tau = \frac{1}{2\omega^2} \left[\frac{1}{2\omega} \sin(2\omega\tau - \omega t) - \tau \cos \omega t \right]_0^t \\ &= \frac{1}{2\omega^2} \left\{ \left[\frac{1}{2\omega} (\sin \omega t - \sin(-\omega t)) \right] - t \cos \omega t \right\} = \frac{1}{2\omega^3} (\sin \omega t - \omega t \cos \omega t) \end{aligned}$$

$$(b) \mathcal{L}^{-1}\left[\frac{1}{\omega} \sin \omega t\right] = \frac{1}{s^2 + \omega^2} \quad \mathcal{L}^{-1}[\cos \omega t] = \frac{s}{s^2 + \omega^2}$$

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{s}{(s^2 + \omega^2)^2}\right] &= \frac{1}{\omega} \int_0^t \sin \omega \tau \cos \omega(t - \tau) d\tau \\ &= \frac{1}{\omega} \int_0^t \frac{1}{2} [\sin(\omega\tau + \omega t - \omega\tau) + \sin(\omega\tau - \omega t + \omega\tau)] d\tau \\ &= \frac{1}{2\omega} \int_0^t [\sin \omega t + \sin(2\omega\tau - \omega t)] d\tau = \frac{1}{2\omega} \left[\tau \sin \omega t + \frac{-1}{2\omega} \cos(2\omega\tau - \omega t) \right]_0^t \\ &= \frac{1}{2\omega} \left\{ t \sin \omega t - \frac{1}{2\omega} [\cos \omega t - \cos(-\omega t)] \right\} = \frac{t}{2\omega} \sin \omega t \end{aligned}$$

II. Partial Fraction

If $F(s) = \frac{P(s)}{Q(s)}$, where $\deg[P(s)] < \deg[Q(s)]$

1. $Q(s) = 0$ with unrepeated factors $s - a_i$

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - a_1} + \frac{A_2}{s - a_2} + \dots + \frac{A_n}{s - a_n}$$

$$A_k = \lim_{s \rightarrow a_k} \left[\frac{P(s)}{Q(s)} (s - a_k) \right] = P(a_k) \lim_{s \rightarrow a_k} \frac{s - a_k}{Q(s)}$$

$$= P(a_k) \lim_{s \rightarrow a_k} \frac{1}{Q'(s)} = \frac{P(a_k)}{Q'(a_k)}$$

$$\frac{P(s)}{Q(s)} = \frac{P(a_1)/Q'(a_1)}{s - a_1} + \frac{P(a_2)/Q'(a_2)}{s - a_2} + \dots + \frac{P(a_n)/Q'(a_n)}{s - a_n}$$

$$\mathcal{L}^{-1} \left[\frac{P(s)}{Q(s)} \right] = \frac{P(a_1)}{Q'(a_1)} e^{a_1 t} + \frac{P(a_2)}{Q'(a_2)} e^{a_2 t} + \dots + \frac{P(a_n)}{Q'(a_n)} e^{a_n t}$$

Question:

$$\mathcal{L}^{-1} \left[\frac{s+1}{s^3 + s^2 - 6s} \right].$$

$$\text{Solution: } \frac{s+1}{s^3 + s^2 - 6s} = \frac{s+1}{s(s-2)(s+3)} = \frac{A_1}{s} + \frac{A_2}{s-2} + \frac{A_3}{s+3}$$

$$A_1 = \lim_{s \rightarrow 0} \frac{s+1}{(s-2)(s+3)} = -\frac{1}{6}$$

$$A_2 = \lim_{s \rightarrow 2} \frac{s+1}{s(s+3)} = \frac{3}{10}$$

$$A_3 = \lim_{s \rightarrow -3} \frac{s+1}{s(s-2)} = \frac{-2}{15}$$

$$\mathcal{L}^{-1} \left[\frac{s+1}{s^3 + s^2 - 6s} \right] = -\frac{1}{6} + \frac{3}{s-2} + \frac{-2}{s+3} = -\frac{1}{6} + \frac{3}{10} e^{2t} - \frac{2}{15} e^{-3t}$$

2. $Q(s) = 0$ with repeated factors $(s - a_k)^m$

$$\frac{P(s)}{Q(s)} = \frac{C_m}{(s - a_k)^m} + \frac{C_{m-1}}{(s - a_k)^{m-1}} + \dots + \frac{C_1}{s - a_k}$$

$$\frac{P(s)}{Q(s)} (s - a_k)^m = C_m + C_{m-1}(s - a_k) + C_{m-2}(s - a_k)^2 + \dots + C_1(s - a_k)^{m-1}$$

$$C_m = \lim_{s \rightarrow a_k} \left[\frac{P(s)}{Q(s)} (s - a_k)^m \right]$$

$$C_{m-1} = \lim_{s \rightarrow a_k} \left\{ \frac{d}{ds} \left[\frac{P(s)}{Q(s)} (s - a_k)^m \right] \right\}$$

$$C_{m-2} = \lim_{s \rightarrow a_k} \left\{ \frac{d^2}{ds^2} \left[\frac{P(s)}{Q(s)} (s - a_k)^m \right] \right\} \frac{1}{2!}$$

.....

$$C_1 = \lim_{s \rightarrow a_k} \left\{ \frac{d^{m-1}}{ds^{m-1}} \left[\frac{P(s)}{Q(s)} (s - a_k)^m \right] \right\} \frac{1}{(m-1)!}$$

$$\mathcal{L}^{-1} \left[\frac{P(s)}{Q(s)} \right] = e^{a_k t} \left[C_m \frac{t^{m-1}}{(m-1)!} + C_{m-1} \frac{t^{m-2}}{(m-2)!} + \dots + C_2 t + C_1 \right]$$

Question:

$$\mathcal{L}^{-1} \left[\frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-1)(s-2)(s-3)} \right].$$

$$\text{Solution: } \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-1)(s-2)(s-3)} = \frac{C_2}{s^2} + \frac{C_1}{s} + \frac{A_1}{s-1} + \frac{A_2}{s-2} + \frac{A_3}{s-3}$$

$$C_2 = \lim_{s \rightarrow 0} \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{(s-1)(s-2)(s-3)} = \frac{-12}{-6} = 2$$

$$\begin{aligned} C_1 &= \lim_{s \rightarrow 0} \frac{d}{ds} \left[\frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{(s-1)(s-2)(s-3)} \right] \\ &= \frac{4(-1)(-2)(-3) - (-12)[(-2)(-3) + (-1)(-3) + (-1)(-2)]}{[(-1)(-2)(-3)]^2} = \frac{-24 + 12 \times 11}{6^2} = 3 \end{aligned}$$

$$A_1 = \lim_{s \rightarrow 1} \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-2)(s-3)} = \frac{-1}{2}$$

$$A_2 = \lim_{s \rightarrow 2} \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-1)(s-3)} = \frac{8}{-4} = -2$$

$$A_3 = \lim_{s \rightarrow 3} \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-1)(s-2)} = \frac{9}{18} = \frac{1}{2}$$

$$\mathcal{L}^{-1} \left[\frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-1)(s-2)(s-3)} \right] = 2t + 3 - \frac{1}{2}e^t - 2e^{2t} + \frac{1}{2}e^{3t}$$

3. $Q(s) = 0$ with unrepeated factor $(s - \alpha)^2 + \beta$, where $\beta > 0$

$$\frac{P(s)}{Q(s)} = \frac{As + B}{(s - \alpha)^2 + \beta^2}$$

$$\frac{P(s)}{Q(s)} [(s - \alpha)^2 + \beta^2] = As + B$$

$$\lim_{s \rightarrow \alpha + i\beta} \left\{ \frac{P(s)}{Q(s)} [(s - \alpha)^2 + \beta^2] \right\} = A(\alpha + i\beta) + B$$

$$R + iI = (A\alpha + \beta) + iA\beta$$

where R and I are the real and imaginary parts of $\lim_{s \rightarrow \alpha + i\beta} \left\{ \frac{P(s)}{Q(s)} [(s - \alpha)^2 + \beta^2] \right\}$, respectively

then, $\begin{cases} A\alpha + B = R \\ A\beta = I \end{cases}$, where we can get A and B , and

$$\mathcal{L}^{-1} \left[\frac{P(s)}{Q(s)} \right] = \mathcal{L}^{-1} \left[\frac{A(s - \alpha) + (A\alpha + B)}{(s - \alpha)^2 + \beta^2} \right] = e^{\alpha t} \left(A \cos \beta t + \frac{A\alpha + B}{\beta} \sin \beta t \right)$$

Question:

$$\mathcal{L}^{-1} \left[\frac{s^2}{s^4 + 4} \right].$$

$$\begin{aligned} \text{Solution: } \frac{s^2}{s^4 + 4} &= \frac{s^2}{(s^2)^2 + 2 \cdot s^2 \cdot 2 + 2^2 - 2 \cdot s^2 \cdot 2} = \frac{s^2}{(s^2 + 2)^2 - (2s)^2} \\ &= \frac{s^2}{(s^2 + 2s + 2)(s^2 - 2s + 2)} = \frac{A_1 s + B_1}{(s + 1)^2 + 1} + \frac{A_2 s + B_2}{(s - 1)^2 + 1} \end{aligned}$$

$$\lim_{s \rightarrow -1+i} \frac{s^2}{(s - 1)^2 + 1} = A_1(-1 + i) + B_1 \Rightarrow \frac{-2i}{4 - 4i} = (-A_1 + B_1) + iA_1$$

$$\frac{8 - 8i}{32} = (-A_1 + B_1) + iA_1 \Rightarrow A_1 = -\frac{1}{4}, B_1 = 0$$

$$\lim_{s \rightarrow 1+i} \frac{s^2}{(s + 1)^2 + 1} = A_2(1 + i) + B_2 \Rightarrow \frac{2i}{4 + 4i} = (A_2 + B_2) + iA_2$$

$$\frac{8 + 8i}{32} = (A_2 + B_2) + iA_2 \Rightarrow A_2 = \frac{1}{4}, B_2 = 0$$

$$\mathcal{L}^{-1} \left[\frac{s^2}{s^4 + 4} \right] = \mathcal{L}^{-1} \left[\frac{-\frac{1}{4}(s + 1) + \frac{1}{4}}{(s + 1)^2 + 1} + \frac{\frac{1}{4}(s - 1) + \frac{1}{4}}{(s - 1)^2 + 1} \right]$$

$$= \frac{e^{-t}}{4} (-\cos t + \sin t) + \frac{e^t}{4} (\cos t + \sin t)$$

4. $Q(s)=0$ with repeated complex factor $[(s-\alpha)^2 + \beta]^2$, where $\beta > 0$

$$\frac{P(s)}{Q(s)} = \frac{As + B}{[(s-\alpha)^2 + \beta]^2} + \frac{Cs + D}{(s-\alpha)^2 + \beta}$$

$$\frac{P(s)}{Q(s)} [(s-\alpha)^2 + \beta]^2 = As + B + (Cs + D)[(s-\alpha)^2 + \beta]$$

$$\lim_{s \rightarrow \alpha + i\beta} \left\{ \frac{P(s)}{Q(s)} [(s-\alpha)^2 + \beta]^2 \right\} = A(\alpha + i\beta) + B$$

$$R_1 + iI_1 = (A\alpha + B) + iA\beta \Rightarrow \begin{cases} A\alpha + B = R_1 \\ A\beta = I_1 \end{cases}, \text{ where } A \text{ and } B \text{ can be obtained}$$

$$\lim_{s \rightarrow \alpha + i\beta} \frac{d}{ds} \left\{ \frac{P(s)}{Q(s)} [(s-\alpha)^2 + \beta]^2 \right\} = A + [C(\alpha + i\beta) + D] \lim_{s \rightarrow \alpha + i\beta} \frac{d}{ds} [(s-\alpha)^2 + \beta]$$

$$R_2 + iI_2 = A + [C(\alpha + i\beta) + D]2i\beta = (A - 2C\beta^2) + i(2\alpha\beta C + 2\beta D)$$

$$\Rightarrow \begin{cases} A - 2C\beta^2 = R_2 \\ 2\alpha\beta C + 2\beta D = I_2 \end{cases}, \text{ where we get } C \text{ and } D, \text{ hence}$$

$$\begin{aligned} \mathcal{L}^{-1} \left[\frac{P(s)}{Q(s)} \right] &= \mathcal{L}^{-1} \left\{ \frac{A(s-\alpha) + (A\alpha + B)}{[(s-\alpha)^2 + \beta]^2} \right\} + \mathcal{L}^{-1} \left[\frac{C(s-\alpha) + (C + D)}{(s-\alpha)^2 + \beta} \right] \\ &= e^{\alpha t} \left\{ \left[\frac{At}{2\beta} \sin \beta t + (A\alpha + B) \frac{1}{2\beta^3} (\sin \beta t - \beta t \cos \beta t) \right] + [C \cos \beta t + (C\alpha + D) \frac{1}{\beta} \sin \beta t] \right\} \end{aligned}$$

Question:

$$\mathcal{L}^{-1} \left[\frac{s^3 - 3s^2 + 6s - 4}{(s^2 - 2s + 2)^2} \right].$$

$$\text{Solution: } \frac{s^3 - 3s^2 + 6s - 4}{(s^2 - 2s + 2)^2} = \frac{As + B}{[(s-1)^2 + 1]^2} + \frac{cs + D}{(s-1)^2 + 1}$$

$$\lim_{s \rightarrow 1+i} (s^3 - 3s^2 + 6s + 4) = A(1+i) + B$$

$$2i = (A + B) + iA \Rightarrow A = 2, B = -2$$

$$\lim_{s \rightarrow 1+i} \frac{d}{ds} (s^3 - 3s^2 + 6s + 4) = A + [c(1+i) + D] \lim_{s \rightarrow 1+i} \frac{d}{ds} [(s-1)^2 + 1]$$

$$0 = A + (c + ic + D)2i = (A - 2c) + 2i(c + D)$$

$$c = 1, D = -1$$

$$\mathcal{L}^{-1} \left[\frac{s^3 - 3s^2 + 6s - 4}{(s^2 - 2s + 2)^2} \right] = \mathcal{L}^{-1} \left\{ \frac{2(s-1)}{[(s-1)^2 + 1]^2} \right\} + \mathcal{L}^{-1} \left[\frac{s-1}{(s-1)^2 + 1} \right]$$

$$= e^t \left(2 \cdot \frac{t}{2} \sin t + \cos t \right) = e^t (t \sin t + \cos t)$$

III. Series Method

$$\mathcal{L}^{-1}\left[\sin \frac{1}{s}\right].$$

$$\text{Solution : } \sin \frac{1}{s} = \frac{1}{s} - \frac{1}{3!} \left(\frac{1}{s}\right)^3 + \frac{1}{5!} \left(\frac{1}{s}\right)^5 - \frac{1}{7!} \left(\frac{1}{s}\right)^7 + \dots$$

$$\begin{aligned} \mathcal{L}^{-1}\left[\sin \frac{1}{s}\right] &= 1 - \frac{t^2}{2!3!} + \frac{t^4}{4!5!} - \frac{t^6}{6!7!} + \dots \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!(2n+1)!} \end{aligned}$$

IV. Differentiation with Respect to a Number

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2 + \omega^2)^2}\right].$$

$$\text{Solution : } \frac{d}{d\omega} \left(\frac{1}{s^2 + \omega^2}\right) = \frac{-2\omega}{(s^2 + \omega^2)^2} \Rightarrow \mathcal{L}^{-1}\left[\frac{d}{d\omega} \left(\frac{1}{s^2 + \omega^2}\right)\right] = \mathcal{L}^{-1}\left[\frac{-2\omega}{(s^2 + \omega^2)^2}\right]$$

$$-2\omega \mathcal{L}^{-1}\left[\frac{1}{(s^2 + \omega^2)^2}\right] = \frac{d}{d\omega} \mathcal{L}^{-1}\left[\frac{1}{s^2 + \omega^2}\right] = \frac{d}{d\omega} \left(\frac{1}{\omega} \sin \omega t\right) = -\frac{1}{\omega^2} \sin \omega t + \frac{t}{\omega} \cos \omega t$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2 + \omega^2)^2}\right] = \frac{1}{2\omega^3} (\sin \omega t - \omega t \cos \omega t)$$

V. Method of Differential Equation

$$\mathcal{L}^{-1}[e^{-\sqrt{s}}].$$

$$\text{Solution : } \bar{y} = e^{-\sqrt{s}} \Rightarrow \bar{y}' = -\frac{e^{-\sqrt{s}}}{2\sqrt{s}}, \bar{y}'' = \frac{e^{-\sqrt{s}}}{4s} + \frac{e^{-\sqrt{s}}}{4\sqrt{s}^3}$$

$$\text{we get the equation } 4s\bar{y}'' + 2\bar{y}' - \bar{y} = 0 \Rightarrow 4\mathcal{L} \left[\frac{d}{dt}(t^2 y) \right] + 2\mathcal{L}[-ty] - \mathcal{L}[y] = 0$$

$$4 \frac{d}{dt}(t^2 y) - 2ty - y = 0 \Rightarrow 4t^2 y' + (6t - 1)y = 0 \Rightarrow \frac{dy}{y} + \frac{6t - 1}{4t^2} dt = 0$$

$$\ln y + \frac{3}{2} \ln t + \frac{1}{4t} = c_1 \Rightarrow y = ct^{-\frac{3}{2}} e^{-\frac{1}{4t}}$$

$$\therefore \mathcal{L} \left[t^{-\frac{1}{2}} \right] = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}}, \text{ and } \mathcal{L}[ty] = \mathcal{L} \left[ct^{-\frac{1}{2}} e^{-\frac{1}{4t}} \right]$$

$$\text{while } \mathcal{L}[ty] = -\bar{y}' = \frac{e^{-\sqrt{s}}}{2\sqrt{s}} \Rightarrow \mathcal{L} \left[ct^{-\frac{1}{2}} e^{-\frac{1}{4t}} \right] = \frac{e^{-\sqrt{s}}}{2\sqrt{s}}$$

$$\text{Apply general final value theorem } \lim_{t \rightarrow \infty} \frac{ct^{-\frac{1}{2}} e^{-\frac{1}{4t}}}{t^{-\frac{1}{2}}} = \lim_{s \rightarrow 0} \frac{\frac{e^{-\sqrt{s}}}{2\sqrt{s}}}{\frac{\sqrt{\pi}}{\sqrt{s}}} \Rightarrow c = \frac{1}{2\sqrt{\pi}}$$

$$\therefore y = \frac{1}{2\sqrt{\pi} t^{3/2}} e^{-\frac{1}{4t}}$$

VI. Apply the Theory of Complex Variables

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s) e^{st} ds$$

4.5. Solve Differential Equations

Question:

$$y'' + y' + y = g(x), \quad y(0) = 1, \quad y'(0) = 0, \quad \text{where } g(x) = \begin{cases} 1 & 0 < x < 3 \\ 3 & x > 3 \end{cases}.$$

Solution : $g(x) = u(x) + 2u(x-3)$

$$[s^2 Y - sy(0) - y'(0)] + [sY - y(0)] + Y = \frac{1}{s} + 2 \frac{e^{-3s}}{s}$$

$$(s^2 + s + 1)Y = s + 1 + \frac{1}{s} + 2 \frac{e^{-3s}}{s}$$

$$Y = \frac{s+1}{s^2+s+1} + \frac{1}{s(s^2+s+1)} + \frac{2e^{-3s}}{s(s^2+s+1)}$$

$$= \frac{s+1}{s^2+s+1} + \left(\frac{1}{s} - \frac{s+1}{s^2+s+1}\right) + 2e^{-3s} \left(\frac{1}{s} - \frac{s+1}{s^2+s+1}\right)$$

$$\frac{s+1}{s^2+s+1} = \frac{(s+\frac{1}{2}) + \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \Rightarrow \mathcal{L}^{-1}\left[\frac{s+1}{s^2+s+1}\right] = e^{-\frac{x}{2}} \left(\cos \frac{\sqrt{3}}{2} x + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} x\right)$$

$$y(x) = u(x) + 2u(x-3) \left\{ 1 - e^{-\frac{x-3}{2}} \left[\cos \frac{\sqrt{3}}{2} (x-3) + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} (x-3) \right] \right\}$$

Question:

$$y'''(t) - 2y''(t) + 5y'(t) = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y\left(\frac{\pi}{8}\right) = 1.$$

Solution : $[s^3 Y - s^2 y(0) - sy'(0) - y''(0)] - 2[s^2 Y - sy(0) - y'(0)] + 5[sY - y(0)] = 0$

$$y''(0) = c$$

$$Y = \frac{s+c-2}{s(s^2-2s+5)} = \frac{A}{s} + \frac{Ps+Q}{(s-1)^2+2^2}$$

$$A = \lim_{s \rightarrow 0} \frac{s+c-2}{s^2-2s+5} = \frac{c-2}{5}$$

$$P(1+2i) + Q = \lim_{s \rightarrow 1+2i} \frac{s+c-2}{s} = \frac{-1+c+2i}{1+2i} = \frac{c+3}{5} + \frac{4-2c}{5}i$$

$$P = \frac{2-c}{5}, \quad Q = \frac{2c+1}{5}$$

$$y(t) = \frac{c-2}{5} + e^t \left(\frac{2-c}{5} \cos 2t + \frac{c+3}{10} \sin 2t \right)$$

$$y\left(\frac{\pi}{8}\right) = 1 \Rightarrow 1 = \frac{c-2}{5} + e^{\frac{\pi}{8}} \left(\frac{2-c}{5} \frac{1}{\sqrt{2}} + \frac{c+3}{10} \frac{1}{\sqrt{2}} \right) \Rightarrow c = 7$$

$$\therefore y(t) = 1 + e^t (-\cos 2t + \sin 2t)$$

II. Ordinary Differential Equations with Variable Coefficients

Question:

$$ty'' + (1 - 2t)y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

$$\text{Solution : } -\frac{d}{ds}[s^2Y - sy(0) - y'(0)] + \{[sY - y(0)] + 2\frac{d}{ds}[sY - y(0)]\} - 2Y = 0$$

$$(-s^2Y' - 2sY + 1) + [(sY - 1) + 2(sY' + Y)] - 2Y = 0$$

$$(-s^2 + 2s)Y' + (-2s + s + 2 - 2)Y = 0$$

$$-(s - 2)Y' = Y \Rightarrow \frac{dY}{Y} = -\frac{ds}{s - 2} \Rightarrow \ln Y = -\ln(s - 2) + c_1$$

$$Y = \frac{c}{s - 2} \Rightarrow y(t) = ce^{2t}$$

$$y(0) = 1, \therefore 1 = c, \quad y(t) = e^{2t}$$

III. Simultaneous Ordinary Differential Equations

Question:

$$\begin{cases} \frac{dx}{dt} = 2x + y + 2e^{5t} \\ \frac{dy}{dt} = x + 2y + 3e^{2t} \end{cases}, \quad x(0) = y(0) = 0.$$

$$\text{Solution : } \begin{cases} sX - x(0) = 2X + Y + \frac{2}{s-5} \\ sY - y(0) = X + 2Y + \frac{3}{s-2} \end{cases} \Rightarrow \begin{cases} (s-2)X - Y = \frac{2}{s-5} \\ -X + (s-2)Y = \frac{3}{s-2} \end{cases}$$

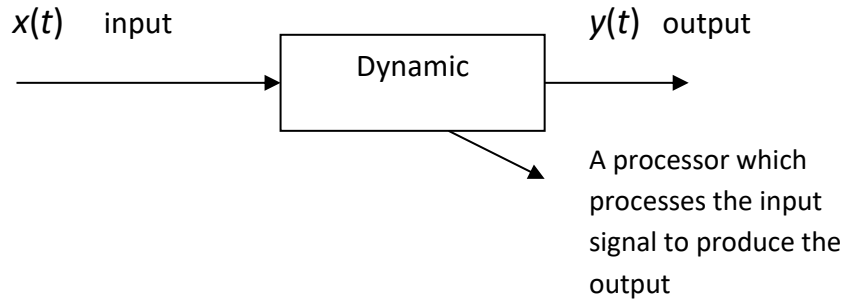
$$X = \frac{(s-2)\frac{2}{s-5} + \frac{3}{s-2}}{(s-2)^2 - 1} = \frac{2s^2 - 5s - 7}{(s-1)(s-2)(s-3)(s-5)}$$

$$Y = \frac{\frac{2}{s-5} + (s-2)\frac{3}{s-2}}{(s-2)^2 - 1} = \frac{3s - 13}{(s-1)(s-3)(s-5)}$$

$$X = \frac{5/4}{s-1} + \frac{-3}{s-2} + \frac{1}{s-3} + \frac{3/4}{s-5} \Rightarrow x(t) = \frac{5}{4}e^t - 3e^{2t} + e^{3t} + \frac{3}{4}e^{5t}$$

$$Y = \frac{-5/4}{s-1} + \frac{1}{s-3} + \frac{1/4}{s-5} \Rightarrow y(t) = -\frac{5}{4}e^t + e^{3t} + \frac{1}{4}e^{5t}$$

4.6. Dynamic Systems



Static system: $y(t) = ax(t) \Rightarrow$ easy (simple processing)

Dynamic system:

$$\frac{dy^{(n)}(t)}{dt^n} + a_1 \frac{dy^{(n-1)}(t)}{dt^{n-1}} + \dots + a_n y(t) = b_0 \frac{dx^{(m)}(t)}{dt^m} + \dots + b_m x(t)$$

Örnek:

$$\frac{d^3 y}{dt^3} + 6 \frac{d^2 y}{dt^2} + 11 \frac{dy}{dt} + 6y = 4 \frac{du}{dt} + 2u$$

$$y(0) = y'(0) = y''(0) = 0$$

$$\frac{du}{dt} = 0 \text{ at } t=0$$

$$s^3 Y(s) + 6s^2 Y(s) + 11s Y(s) + 6Y(s) = 4sU(s) + 2U(s)$$

$$Y(s) = \frac{4s+2}{s^3 + 6s^2 + 11s + 6} \cdot \frac{1}{s}$$

$$U(s) = \frac{1}{s} \text{ (unit step input)}$$

$$s(s^3 + 6s^2 + 11s + 6) = s(s+1)(s+2)(s+3)$$

$$Y(s) = \frac{1}{3s} + \frac{1}{s+1} - \frac{3}{s+2} + \frac{5/3}{s+3}$$

$$\frac{4s+2}{s(s+1)(s+2)(s+3)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+1} + \frac{\alpha_3}{s+2} + \frac{\alpha_4}{s+3}$$

$$y(t) = \frac{1}{3} + e^{-t} - 3e^{-2t} + \frac{5}{3}e^{-3t}$$

$$t \rightarrow \infty \quad y(t) \rightarrow \frac{1}{3}$$

$$1. \mathcal{L} [1] = \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}$$

$$2. \mathcal{L} [t^a] = \int_0^{\infty} t^a e^{-st} dt = \int_0^{\infty} \left(\frac{u}{s}\right)^a e^{-u} \frac{du}{s} = \frac{1}{s^{a+1}} \int_0^{\infty} u^a e^{-u} du = \frac{\Gamma(a+1)}{s^{a+1}}$$

$$3. \mathcal{L} [e^{at}] = \int_0^{\infty} e^{at} e^{-st} dt = \frac{e^{-(s-a)t}}{-(s-a)} \Big|_0^{\infty} = \frac{1}{s-a}$$

$$4. \mathcal{L} [e^{iat}] = \frac{1}{s-ia} \Rightarrow \mathcal{L} [\cos at + i \sin at] = \frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2}$$

$$\therefore \mathcal{L} [\cos at] = \frac{s}{s^2+a^2}, \text{ and } \mathcal{L} [\sin at] = \frac{a}{s^2+a^2}$$

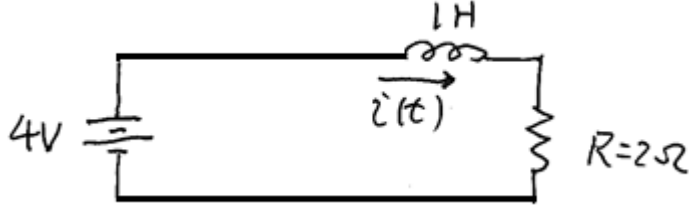
$$5. \mathcal{L} [\sinh at] = \mathcal{L} \left[\frac{e^{at} - e^{-at}}{2} \right] = \frac{1}{2} \left(\frac{1}{s-a} - \frac{1}{s+a} \right) = \frac{a}{s^2-a^2}$$

$$\mathcal{L} [\cosh at] = \mathcal{L} \left[\frac{e^{at} + e^{-at}}{2} \right] = \frac{1}{2} \left(\frac{1}{s-a} + \frac{1}{s+a} \right) = \frac{s}{s^2-a^2}$$

4.7. Uygulamalar

Soru:

$t > 0$ için Laplace dönüşümünü kullanarak aşağıda verilmiş olan $i(t)$ akım değeri bulunacaktır.



- a) Kirchhoff Gerilim Yasasına göre aşağıdaki denklem yazıldığından devrede kullanılan eleman değerlerini yerine koyarak denklemi düzenleyiniz.

$$L \frac{di(t)}{dt} + Ri(t) = 0$$

$$\frac{di(t)}{dt} + 2i(t) = 0$$

- b) Düzenlediğiniz diferansiyel denklemi Laplace dönüşümünü yazınız. Toplamın Laplace dönüşümü toplamı oluşturan bileşenlerin Laplace dönüşümlerinin toplamına eşittir.

$$L \left[\frac{di(t)}{dt} + 2i(t) \right] = L \left[\frac{di(t)}{dt} \right] + 2L[i(t)] = 0$$

$$L \left[\frac{di(t)}{dt} \right] = sI(s) - i(0^-)$$

$$L[i(t)] = I(s)$$

$$sI(s) - i(0^-) + 2I(s) = 0$$

- c)

$$i(0^-) = 2$$

olduğuna göre $I(s)$ ifadesini “ s ” cinsinden bulunuz.

$$(s + 2)I(s) - 2 = 0$$

$$I(s) = \frac{2}{s+2}$$

d) Ters Laplace dönüşümünden $i(t)$ ifadesini bulunuz.

$$L[e^{-\alpha t}] = \frac{1}{s + \alpha}$$

$$i(t) = 2e^{-2t}u(t) \quad A$$

Soru:

Aşağıdaki Laplace dönüşümü verilen $x(s)$ fonksiyonun Ters Laplace dönüşümünü bulunuz.

$$X(s) = \frac{1}{s-b} + \frac{1}{s+a}, \quad -a < \operatorname{Re}\{s\} < b, \quad -a < b.$$

$$x(t) = e^{-at}u(t) - be^{bt}u(-t)$$

The unilateral Laplace transform of $f(t)$ is $\frac{1}{s^2 + s + 1}$. The unilateral Laplace transform of $tf(t)$ is

Using s -domain differentiation property of Laplace transform.

If $f(t) \xleftrightarrow{\mathcal{L}} F(s)$

$$tf(t) \xleftrightarrow{\mathcal{L}} -\frac{dF(s)}{ds}$$

So, $\mathcal{L}[tf(t)] = \frac{-d}{ds} \left[\frac{1}{s^2 + s + 1} \right] = \frac{2s + 1}{(s^2 + s + 1)^2}$

Soru:

Kirchhoff'un akım yasasına göre bir devrede aşağıdaki denklem yazılmıştır.

$$i_0(t) = i_1(t) + i_2(t).$$

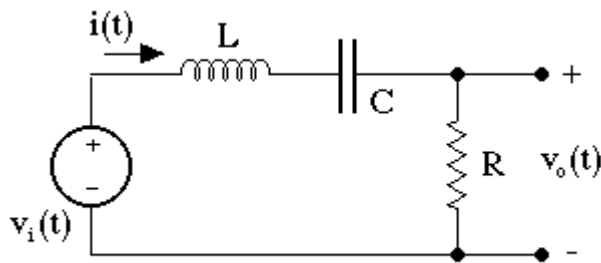
$$i_1(t) = C \frac{dy(t)}{dt} \quad \text{ve} \quad i_2(t) = \frac{y(t)}{R_2}.$$

Denklemleriyle verildiğine göre $i_0(t)$ denkleminde yerine koyunuz, $y(t)$ gerilim sinyali

olduğuna göre $\frac{di_0(t)}{dt}$ yi bulunuz.

Example: RLC Circuits

An Example of the Application of Laplace Transforms



Kirchoff's voltage law

$$v_1(t) = i(t) R + L \frac{di(t)}{dt} + \frac{1}{C} \int_{0_-}^t i(t') dt'$$

Kirchoff's voltage law transformed

$$V_1(s) = R I(s) + s L I(s) - L i(0_-) + \frac{1}{s C} I(s)$$

$$= \left[R + s L + \frac{1}{s C} \right] I(s) - L i(0_-)$$

$$= Z(s) I(s) - L i(0_-)$$

$$Z(s) = R + s L + \frac{1}{s C}$$

where

Example 1 - A series RLC circuit excited by a unit impulse function - that is to say

$$V_1(s) = 1$$

(The unit impulse always gives the "natural response" of any circuit. Natural as compared to "forced" response.)

$$I(s) = \frac{1}{Z(s)} = \frac{1}{R + sL + \frac{1}{sC}} = \frac{1}{L} \left\{ \frac{s}{s^2 + s \frac{R}{L} + \frac{1}{LC}} \right\}$$

This is to be compared to the transform pair

$$f(t) = \sqrt{c^2 + d^2} \exp(-at) \cos\left(\omega t - \tan^{-1} \frac{d}{c}\right)$$

$$F(s) = \frac{c(s+a) + d\omega}{(s+a)^2 + \omega^2}$$

$$a = \frac{R}{2L} \Rightarrow c = 1 \Rightarrow d = -\frac{\omega}{a} = -\frac{2\omega L}{R}$$

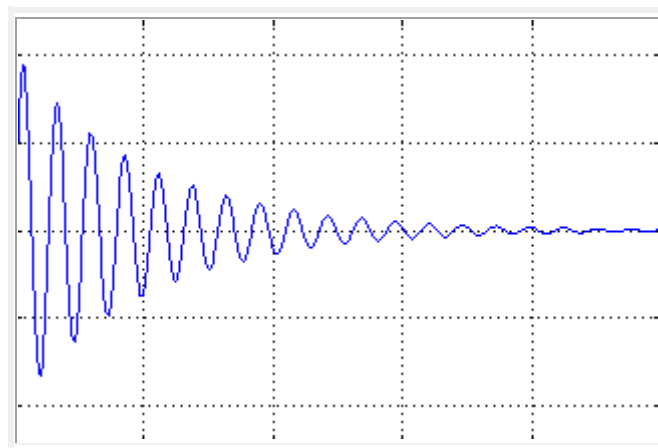
$$\omega^2 = \frac{1}{LC} - a^2 = \frac{1}{LC} - \left(\frac{R}{2L}\right)^2$$

$$\sqrt{c^2 + d^2} = \frac{\sqrt{a^2 + \omega^2}}{a} = \frac{2L}{R} \sqrt{\frac{1}{LC}}$$

$$\tan^{-1} \frac{d}{c} = -\tan^{-1} \left(\frac{2\omega L}{R}\right)$$

$$v_2(t) = i(t) R = 2 \sqrt{\frac{1}{LC}} \exp\left(-\frac{R}{2L} t\right) \cos\left\{\omega t + \tan^{-1} \left(\frac{2\omega L}{R}\right)\right\}$$

where $\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$



Örnek:

The [Laplace Transform](#) of $f(t)$ is given by, $F(s) = \frac{8}{s(s+4)}$ Find the final value of equation using final value theorem as well as conventional method of finding final value.

$$\text{Here, } \mathcal{L}f(t) = \frac{8}{s(s+4)}$$

$$\therefore f(\infty) = \lim_{s \rightarrow 0} s \mathcal{L}f(t) = \lim_{s \rightarrow 0} \frac{8s}{s(s+4)} = \lim_{s \rightarrow 0} \frac{8}{s+4} = \frac{8}{4} = 2$$

$$\text{Again, } F(s) = \frac{8}{s(s+4)}$$

That can be written as,

$$F(s) = \frac{k_1}{s} + \frac{k_2}{s+4} \quad [\text{Where, } k_1 \text{ \& } k_2 \text{ are two arbitrary constant}]$$

$$= \frac{sk_1 + 4k_1 + sk_2}{s(s+4)} = \frac{s(k_1 + k_2) + 4k_1}{s(s+4)}$$

$$\therefore 8 = s(k_1 + k_2) + 4k_1$$

$$\Rightarrow k_1 + k_2 = 0 \text{ \& } 4k_1 = 8 \Rightarrow k_1 = 2 \text{ \& } k_2 = -2$$

$$\text{Now, } F(s) = \frac{2}{s} - \frac{2}{s+4}$$

$$\therefore \mathcal{L}^{-1}F(s) = 2 - 2e^{-4t} = f(t)$$

$$\therefore f(\infty) = \lim_{t \rightarrow \infty} (2 - 2e^{-4t}) = 2 - 2e^{-\infty} = 2$$

Örnek:

Find the Inverse [Laplace Transformation](#) of function, $F(s) = \frac{2}{(s+2)(s+8)}$

$$F(s) = \frac{2}{(s+2)(s+8)} = \frac{k_1}{s+2} + \frac{k_2}{s+8}$$
$$= \frac{k_1 s + 8k_1 + k_2 s + 2k_2}{(s+2)(s+8)} = \frac{(k_1 + k_2)s + 2(k_2 + 4k_1)}{(s+2)(s+8)}$$

$$\therefore (k_1 + k_2)s + 2(k_2 + 4k_1) = 2$$

$$\Rightarrow (k_1 + k_2)s + 2(k_2 + 4k_1) = 0.s + 2.1$$

$$k_1 + k_2 = 0 \ \& \ 2(k_2 + 4k_1) = 1 \Rightarrow k_1 = \frac{1}{3} \ \& \ k_2 = -\frac{1}{3}$$

$$\therefore F(s) = \frac{1}{3} \times \frac{1}{s+2} - \frac{1}{3} \times \frac{1}{s+8}$$

$$\therefore \mathcal{L}F(s) = \frac{1}{3}e^{-2t} - \frac{1}{3}e^{-8t} = \frac{1}{3}(e^{-2t} - e^{-8t})$$

Örnek:

Find the Inverse Laplace transformation of $F(s) = \frac{1}{s(s+1)(s+2)}$

$$F(s) = \frac{1}{s(s+1)(s+2)} = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

[Where, k_1 , k_2 & k_3 are arbitrary constants]

By solving this equation we get,

$$k_1 = \frac{1}{2}, k_2 = -1, \text{ \& } k_3 = \frac{1}{2}$$

$$\therefore F(s) = \frac{1}{2} \times \frac{1}{s} - \frac{1}{s+1} + \frac{1}{2} \times \frac{1}{s+2}$$

F(s) can be rewritten as, $\mathcal{L}F(s) = f(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$

Örnek:

Find the Inverse Laplace transformation of $F(s) = \frac{2}{s(s+1)(s+2)}$

$$F(s) = \frac{2}{s(s+1)(s+2)} = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

[Where, k_1 , k_2 & k_3 are arbitrary constants]

By solving this equation we get,

$$k_1 = 1, k_2 = -2, \text{ \& } k_3 = 1$$

$$\therefore F(s) = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

F(s) can be rewritten as, $\mathcal{L}F(s) = f(t) = 1 - 2e^{-t} + e^{-2t}$

Örnek:

Express the differential equation in Laplace transformation form

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0 \text{ given } x(0^+) = 0 \text{ and } x'(0^+) = 2$$

$$\mathcal{L} \left[\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x \right] = \mathcal{L}0$$

$$\Rightarrow \mathcal{L} \left[\frac{d^2x}{dt^2} \right] + 6\mathcal{L} \left[\frac{dx}{dt} \right] + 8\mathcal{L}[x] = 0$$

$$\Rightarrow s^2X(s) - s x(0^+) - x'(0^+) + 6 \{sX(s) - x(0^+)\} + 8X(s) = 0$$

$$\Rightarrow s^2X(s) - s \cdot 0 - 2 + 6 \{sX(s) - 0\} + 8X(s) = 0$$

$$\Rightarrow X(s) [s^2 + 6s + 8] = 2 \Rightarrow X(s) = \frac{2}{s^2 + 6s + 8}$$

Örnek:

Express the differential equation in [Laplace transformation](#) form

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 1 \text{ given } x(0^+) = 0 \text{ and } x'(0^+) = 0$$

$$\mathcal{L} \left[\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x \right] = \mathcal{L}(1)$$

$$\Rightarrow \mathcal{L} \left[\frac{d^2x}{dt^2} \right] + 4\mathcal{L} \left[\frac{dx}{dt} \right] + \mathcal{L}[x] = \mathcal{L}(1)$$

$$\Rightarrow s^2\mathcal{L}x - sx(0^+) - x'(0^+) + 4 \{s\mathcal{L}x - x(0^+)\} + \mathcal{L}x = \mathcal{L}(1)$$

$$\Rightarrow s^2X(s) + 4sX(s) + X(s) = \frac{1}{s}$$

[As all initial conditions are zero]

$$\therefore X(s) = \frac{1}{s(s^2 + 4s + 1)}$$

Örnek:

Where, $F(s)$ is the Laplace form of a time domain function $f(t)$. Find the expiration of $f(t)$.

$$F(s) = \frac{1}{s(s+1)}$$

$$\mathcal{L}^{-1}F(s) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$\Rightarrow f(t) = 1 - e^{-t}$$

$$\text{Now, } F(s) = \frac{1}{s(s+1)} = \frac{s+1-s}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

Inverse Laplace Transformation of $F(s)$, is

$$F(s) = \frac{6s^2 + 10s + 2}{s^3 + 3s^2 + 2s}$$

Örnek:

Find Inverse Laplace Transformation function of

$$F(s) = \frac{6s^2 + 10s + 2}{s^3 + 3s^2 + 2s} = \frac{6s^2 + 10s + 2}{s(s^2 + 3s + 2)}$$

$$= \frac{6s^2 + 10s + 2}{s(s^2 + 2s + s + 2)} = \frac{6s^2 + 10s + 2}{s(s+1)(s+2)}$$

$$\frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2} = \frac{6s^2 + 10s + 2}{s(s+1)(s+2)}$$

$$\frac{k_1(s+1)(s+2) + k_2(s+2)s + k_3(s+1)s}{s(s+1)(s+2)} = \frac{6s^2 + 10s + 2}{s(s+1)(s+2)}$$

$$k_1(s^2 + s + 2s + 2) + k_2(s^2 + 2s) + k_3(s^2 + s) = 6s^2 + 10s + 2$$

$$(k_1 + k_2 + k_3)s^2 + (3k_1 + 2k_2 + k_3)s + 2k_1 = 6s^2 + 10s + 2$$

$$\therefore 2k_1 = 2 \Rightarrow k_1 = 1, 1 + k_2 + k_3 = 6 \ \& \ 3 + 2k_2 + k_3 = 10$$

$$\Rightarrow k_2 + k_3 = 5 \ \& \ 2k_2 + k_3 = 7$$

$$\Rightarrow (2k_2 + k_3) - (k_2 + k_3) = 7 - 5$$

$$\Rightarrow k_2 = 2$$

$$\therefore 1 + 2 + k_3 = 6 \Rightarrow k_3 = 3$$

$$F(s) = \frac{6s^2 + 10s + 2}{s^3 + 2s^2 + 2s} = \frac{1}{s} + \frac{2}{s+1} + \frac{1}{s+2}$$

$$\therefore \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left(\frac{1}{s}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + 3\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$$

$$f(t) = 1 + 2e^{-t} + 3e^{-2t}$$

Örnek:

Solve the differential equation

$$\frac{dx}{dt} + 9x = 12 \text{ where } x(0^+) = 1$$

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = s\mathcal{L}f(t) - f(0^+)$$

$$\therefore \mathcal{L}\left[\frac{dx}{dt}\right] = s\mathcal{L}(x) - x(0^+)$$

$$\mathcal{L}[9x] = 9\mathcal{L}(x)$$

$$\text{Now, } \mathcal{L}\left[\frac{dx}{dt} + 9x\right] = \mathcal{L}(12)$$

$$\Rightarrow \mathcal{L}\left[\frac{dx}{dt}\right] + 9\mathcal{L}(x) = \mathcal{L}(12)$$

$$s\mathcal{L}(x) - x(0^+) + 9\mathcal{L}(x) = \frac{12}{s}$$

$$\Rightarrow (s+9)\mathcal{L}(x) = \frac{12}{s} + 1$$

$$\Rightarrow \mathcal{L}(x) = \frac{12+s}{s(s+9)} = \frac{k_1}{s} + \frac{k_2}{s+9}$$

[k_1 & k_2 are two constant]

$$\Rightarrow k_1(s+9) + k_2s = 12 + s$$

$$\Rightarrow (k_1 + k_2)s + 9k_1 = 12 + s$$

$$\Rightarrow k_1 + k_2 = 1 \quad \& \quad k_1 = \frac{12}{9} = \frac{4}{3}$$

$$\Rightarrow k_2 = 1 - \frac{12}{9} = \frac{9-12}{9} = \frac{3}{9} = \frac{1}{3}$$

$$\therefore \frac{4}{3s} + \frac{1}{3(s+9)} = \mathcal{L}(x)$$

$$\therefore x(t) = \frac{4}{3} \mathcal{L}^{-1} \left(\frac{1}{s} \right) + \frac{1}{3} \mathcal{L}^{-1} \left(\frac{1}{s+9} \right)$$

$$\Rightarrow x(t) = \frac{4}{3} + \frac{1}{3} e^{-9t}$$

4) Solve the differential equation,

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0 \text{ given } x(0^+) = 0 \text{ and } x'(0^+) = 1$$

$$\mathcal{L} \frac{d^2f(t)}{dt^2} = s^2 \mathcal{L}f(t) - sf(0^+) - f'(0^+)$$

$$\therefore \mathcal{L} \frac{d^2x}{dt^2} = s^2 \mathcal{L}(x) - sx(0^+) - x'(0^+)$$

$$= s^2 \mathcal{L}(x) - s \cdot 0 - 1 = s^2 \mathcal{L}(x) - 1$$

$$\text{Again, } \mathcal{L} \left[\frac{df(t)}{dt} \right] = s \mathcal{L}f(t) - f(0^+)$$

$$\therefore \mathcal{L} \left[\frac{dx}{dt} \right] = s \mathcal{L}(x) - x(0^+) = s \mathcal{L}(x) - 0 = s \mathcal{L}(x)$$

$$\text{Therefore, } \mathcal{L} \left[\frac{d^2x}{dt^2} + \frac{dx}{dt} + x \right] = \mathcal{L}(0)$$

$$\mathcal{L} \left[\frac{d^2x}{dt^2} \right] + \mathcal{L} \left[\frac{dx}{dt} \right] + \mathcal{L}[x] = 0$$

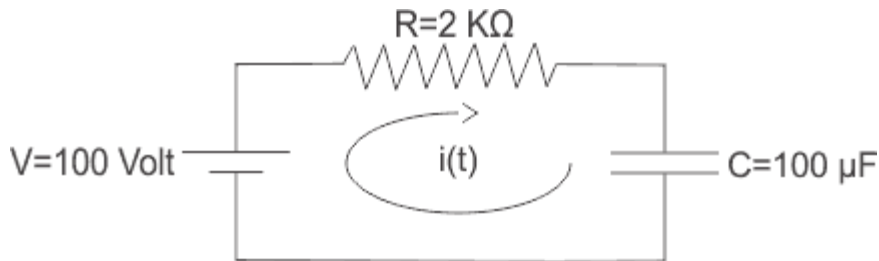
$$\Rightarrow s^2 \mathcal{L}(x) - 1 + s \mathcal{L}(x) + \mathcal{L}(x) = 0$$

$$\Rightarrow \mathcal{L}(x) [s^2 + s + 1] = 1$$

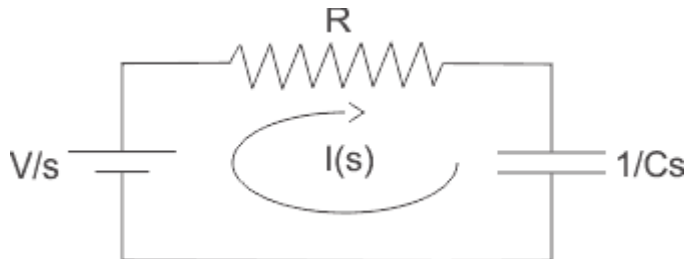
$$\Rightarrow \mathcal{L}(x) = \frac{1}{s^2 + s + 1}$$

$$\begin{aligned} \Rightarrow \mathcal{L}(x) &= \frac{1}{s^2 + 2\frac{1}{2}s + \frac{1}{4} + \frac{3}{4}} \\ &= \frac{1}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \\ &= \frac{2}{\sqrt{3}} \times \frac{(\frac{\sqrt{3}}{2})}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \\ \therefore \mathcal{L}^{-1} \mathcal{L}(x) = x &= \frac{2}{\sqrt{3}} e^{-0.5t} \sin \frac{\sqrt{3}}{2}t \end{aligned}$$

5) For circuit below, calculate the initial charging current of capacitor using Laplace Transform



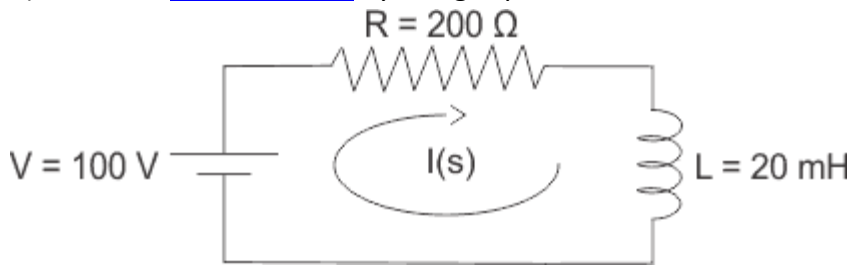
The above figure can be redrawn in Laplace form,



$$\begin{aligned} \text{Therefore, } I(s) &= \frac{\frac{V}{s}}{R + \frac{1}{Cs}} = \frac{\frac{V}{s}Cs}{RCs + 1} = \frac{VC}{RCs + 1} \\ &= \frac{100 \times 100 \times 10^{-6}}{2000 \times 100 \times 10^{-6}s + 1} \\ &= \frac{10^{-2}}{0.2s + 1} \end{aligned}$$

$$\begin{aligned} i(0^+) &= \lim_{t \rightarrow 0} i(t) = \lim_{s \rightarrow \infty} sI(s) = \lim_{s \rightarrow \infty} \frac{s \cdot 10^{-2}}{0.2s + 1} \\ &= \lim_{\frac{1}{s} \rightarrow 0} \frac{10^{-2}}{0.2 + \frac{1}{s}} = \frac{10^{-2}}{0.2} = 0.05 \text{ A} = 50 \text{ mA} \end{aligned}$$

6) Solve the [electric circuit](#) by using Laplace transformation for final steady-state current



Answer

The above circuit can be analyzed by using [Kirchhoff Voltage Law](#) and then we get

$$100 = 100 \times i(t) + 20 \times 10^{-3} \times \frac{di(t)}{dt}$$

$$\Leftrightarrow \mathcal{L}(100) = 100\mathcal{L}[i(t)] + 20 \times 10^{-3} \times \mathcal{L}\left[\frac{di(t)}{dt}\right]$$

$$\Rightarrow \frac{100}{s} = 100 \times I(s) + 20 \times 10^{-3} \times sI(s) \quad [As, i(0^+) = 0]$$

$$\Rightarrow I(s) = \frac{100/s}{100 + 20 \times 10^{-3} \times s}$$

Final value of steady-

$$i(\infty) = \lim_{t \rightarrow \infty} = \lim_{s \rightarrow 0} sI(s) = \lim_{s \rightarrow 0} s \frac{100/s}{100 + s20 \times 10^{-3}}$$

$$= \lim_{s \rightarrow 0} \frac{100}{100 + s20 \times 10^{-3}} = \lim_{s \rightarrow 0} \frac{100}{100} = 1 \text{ A}$$

state current is

7) A system is

represented by the relation $X(s) = R(s) \frac{120}{s^2 + 5s + 10}$ Where, R(s) is the Laplace form of unit

step function. Find the value of x(t) at $t \rightarrow \infty$.

As R(s) is the Laplace form of unit step function, it can be written as

$$R(s) = \frac{1}{s}$$

$$\therefore X(s) = \frac{1}{s} \times \frac{120}{s^2 + 5s + 10}$$

For final value,

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s) = \lim_{s \rightarrow 0} s \times \frac{1}{s} \times \frac{120}{s^2 + 5s + 10}$$

$$= \frac{120}{10} = 12 \text{ units}$$

8) Find $f(t)$, $f'(t)$ and $f''(t)$ for a time domain function $f(t)$. The Laplace Transformation form of

the function is given as $\mathcal{L}f(t) = F(s) = \frac{8s + 3}{2s(s^2 + 5)}$ By applying initial value theorem, we

$$\begin{aligned} f(0^+) &= \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s) \\ &= \lim_{s \rightarrow \infty} \frac{s(8s + 3)}{2s(s^2 + 5)} = \lim_{s \rightarrow \infty} \frac{8/s + 3/s^2}{2 + 10/s^2} \\ &= \lim_{\frac{1}{s} \rightarrow 0} \frac{8/s + 3/s^2}{2 + 10/s^2} = 0 \end{aligned}$$

Now, $\mathcal{L}f'(t) = sF(s) - f(0^+) = sF(s) - 0$

$$= sF(s) \text{ [From, above prove]}$$

$$\therefore f'(0^+) = \lim_{s \rightarrow \infty} sF(s)$$

$$\begin{aligned} &= \lim_{s \rightarrow \infty} \frac{s^2(8s + 3)}{2s(s^2 + 5)} \\ &= \lim_{s \rightarrow \infty} \frac{8 + 3/s}{2 + 10/s^2} = 4 \text{ units} \end{aligned}$$

Now, $\mathcal{L}f''(t) = s^2\mathcal{L}f(t) - sf(0^+) - f'(0^+)$

$$= s^2\mathcal{L}f(t) - s \cdot 0 - 4 = s^2\mathcal{L}f(t) - 4$$

$$\text{Now, } \mathcal{L}f(t) = \frac{8s + 3}{2s(s^2 + 5)}$$

$$\therefore \mathcal{L}f''(t) = \frac{s^2(8s + 3)}{2s(s^2 + 5)} - 4$$

Applying Initial Value Theorem, we

$$\begin{aligned} f''(0^+) &= \lim_{s \rightarrow \infty} s\mathcal{L}f''(t) = \lim_{s \rightarrow \infty} \left[\frac{s^3(8s + 3)}{2s(s^2 + 5)} - 4s \right] \\ \text{get, } &= \lim_{s \rightarrow \infty} \left[\frac{3s^3 - 40s^2}{2s(s^2 + 5)} \right] = \lim_{s \rightarrow \infty} \left[\frac{3 - 40/s}{2 + 10/s^2} \right] = 1.5 \text{ unit} \end{aligned}$$

5. Z-Transformation

5.1. Properties of the Z- transform

For the following

$$Z\{f[n]\} = \sum_{n=0}^{n=\infty} f[n]z^{-n} = F(z)$$

$$Z\{g_n\} = \sum_{n=0}^{n=\infty} g_n z^{-n} = G(z)$$

- **Linearity:**

$Z\{af_n + bg_n\} = aF(z) + bG(z)$. and ROC is $R_f \cap R_g$
which follows from definition of z-transform.

- **Time Shifting**

If we have $f[n] \Leftrightarrow F(z)$ then $f[n-n_0] \Leftrightarrow z^{-n_0} F(z)$

The ROC of $Y(z)$ is the same as $F(z)$ except that there are possible pole additions or deletions at $z = 0$ or $z = \infty$.

Let $y[n] = f[n-n_0]$ then

$$Y(z) = \sum_{n=-\infty}^{\infty} f[n-n_0]z^{-n}$$

Assume $k = n - n_0$ then $n = k + n_0$, substituting in the above equation we have:

$$Y(z) = \sum_{k=-\infty}^{\infty} f[k]z^{-k-n_0} = z^{-n_0} F[z]$$

- **Multiplication by an Exponential Sequence**

Let $y[n] = z_0^n f[n]$ then $Y(z) = X\left(\frac{z}{z_0}\right)$

The consequence is pole and zero locations are scaled by z_0 . If the ROC of $FX(z)$ is $r_R < |z| < r_L$, then the ROC of $Y(z)$ is

$$r_R < |z/z_0| < r_L, \text{ i.e., } |z_0|r_R < |z| < |z_0|r_L$$

$$Y(z) = \sum_{n=-\infty}^{\infty} z_0^n x[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{z}{z_0}\right)^{-n} = X\left(\frac{z}{z_0}\right)$$

The consequence is pole and zero locations are scaled by z_0 . If the ROC of $X(z)$ is $r_R < |z| < r_L$, then the ROC of $Y(z)$ is

$$r_R < |z/z_0| < r_L, \text{ i.e., } |z_0|r_R < |z| < |z_0|r_L$$

- **Differentiation of $X(z)$**

If we have $f[n] \leftrightarrow F(z)$ then $nf[n] \xleftrightarrow{z} -z \frac{dF(z)}{dz}$ and $\text{ROC} = R_f$

$$F(z) = \sum_{n=-\infty}^{\infty} f[n]z^{-n}$$

$$-z \frac{dF(z)}{dz} = -z \sum_{n=-\infty}^{\infty} -n f[n]z^{-n-1} = \sum_{n=-\infty}^{\infty} -n f[n]z^{-n}$$

$$-z \frac{dF(z)}{dz} \xleftrightarrow{z} nf[n]$$

- **Conjugation of a Complex Sequence**

If we have $f[n] \leftrightarrow F(z)$ then $f^*[n] \xleftrightarrow{z} F^*(z^*)$ and $\text{ROC} = R_f$

Proof:

Let $y[n] = f^*[n]$, then

$$Y(z) = \sum_{n=-\infty}^{\infty} f^*[n]z^{-n} = \left(\sum_{n=-\infty}^{\infty} f[n][z^*]^{-n} \right)^* = F^*(z^*)$$

- **Time Reversal**

If we have $f[n] \leftrightarrow F(z)$ then $f^*[-n] \xleftrightarrow{z} F^*(1/z^*)$

Let $y[n] = f^*[-n]$, then

$$Y(z) = \sum_{n=-\infty}^{\infty} f^*[-n]z^{-n} = \left(\sum_{n=-\infty}^{\infty} f[-n][z^*]^{-n} \right)^* = \left(\sum_{k=-\infty}^{\infty} f[k](1/z^*)^{-k} \right)^* = F^*(1/z^*)$$

If the

ROC of $F(z)$ is $r_R < |z| < r_L$, then the ROC of $Y(z)$ is

$$r_R < |1/z^*| < r_L \quad \text{i.e.,} \quad \frac{1}{r_R} > |z| > \frac{1}{r_L}$$

When the time reversal is without conjugation, it is easy to show

$$f[-n] \xleftrightarrow{z} F(1/z) \quad \text{and ROC is} \quad \frac{1}{r_R} > |z| > \frac{1}{r_L}$$

A comprehensive summary for the z-transform properties is shown in Table 2

Table: Summary of z-transform properties

Property	Sequence	z-Transform	Region of Convergence
Linearity	$ax(n) + by(n)$	$aX(z) + bY(z)$	Contains $R_x \cap R_y$
Shift	$x(n - n_0)$	$z^{-n_0}X(z)$	R_x
Time reversal	$x(-n)$	$X(z^{-1})$	$1/R_x$
Exponentiation	$\alpha^n x(n)$	$X(\alpha^{-1}z)$	$ \alpha R_x$
Convolution	$x(n) * y(n)$	$X(z)Y(z)$	Contains $R_x \cap R_y$
Conjugation	$x^*(n)$	$X^*(z^*)$	R_x
Derivative	$nx(n)$	$-z \frac{dX(z)}{dz}$	R_x

Note: Given the z-transforms $X(z)$ and $Y(z)$ of $x(n)$ and $y(n)$, with regions of convergence R_x and R_y , respectively, this table lists the z-transforms of sequences that are formed from $x(n)$ and $y(n)$.

Example: Find the z transform of $3n + 2 \times 3^n$.

From the linearity property

$$Z\{3n + 2 \times 3^n\} = 3Z\{n\} + 2Z\{3^n\}$$

and from the Table 1

$$Z\{n\} = \frac{z}{(z-1)^2} \quad \text{and} \quad Z\{3^n\} = \frac{z}{(z-3)}$$

(r^n with $r = 3$). Therefore

$$Z\{3n + 2 \times 3^n\} = \frac{3z}{(z-1)^2} + \frac{2z}{(z-3)}$$

Example: Find the z-transform of each of the following sequences:

(a) $x(n) = 2^n u(n) + 3(\frac{1}{2})^n u(n)$

(b) $x(n) = \cos(n\omega_0)u(n)$.

(a) Because $x(n)$ is a sum of two sequences of the form $\alpha^n u(n)$, using the linearity property of the z-transform, and referring to Table 1, the z-transform pair

$$X(z) = \frac{1}{1-2z^{-1}} + \frac{3}{1-\frac{1}{2}z^{-1}} = \frac{4 - \frac{13}{2}z^{-1}}{(1-2z)\left(1-\frac{1}{2}z^{-1}\right)}$$

(b) For this sequence we write

$$x(n) = \cos(n\omega_0) u(n) = \frac{1}{2}(e^{jn\omega_0} + e^{-jn\omega_0}) u(n)$$

Therefore, the z-transform is

$$X(z) = \frac{1}{2} \frac{1}{1 - e^{jn\omega_0} z^{-1}} + \frac{1}{2} \frac{1}{1 - e^{-jn\omega_0} z^{-1}}$$

with a region of convergence $|z| > 1$. Combining the two terms together, we have

$$X(z) = \frac{1 - (\cos \omega_0) z^{-1}}{1 - 2(\cos \omega_0) z^{-1} + z^{-2}}$$

5.2. The Inverse z-Transform

The z-transform is a useful tool in linear systems analysis. However, just as important as techniques for finding the z-transform of a sequence are methods that may be used to invert the z-transform and recover the sequence $\mathbf{x}(n)$ from $\mathbf{X}(z)$. Three possible approaches are described below.

- **Partial Fraction Expansion**

For z-transforms that are rational functions of z,

$$X(z) = \frac{\sum_{k=0}^q b(k)z^{-k}}{\sum_{k=0}^p a(k)z^{-k}} = C \frac{\prod_{k=1}^q (1 - \beta_k z^{-1})}{\prod_{k=1}^p (1 - \alpha_k z^{-1})}$$

a simple and straightforward approach to find the inverse z-transform is to perform a partial fraction expansion of $X(z)$. Assuming that $p > q$, and that all of the roots in the denominator are simple, $\alpha_i \neq \alpha_k$ for $i \neq k$, $X(z)$ may be expanded as follows:

$$X(z) = \sum_{k=1}^p \frac{A_k}{1 - \alpha_k z^{-1}} \tag{Eq(3)}$$

for some constants A_k for $k = 1, 2, \dots, p$. The coefficients A_k may be found by multiplying both sides of Eq. (3) by $(1 - \alpha_k z^{-1})$ and setting $z = \alpha_k$. The result is

$$A_k = [(1 - \alpha_k z^{-1})X(z)]_{z=\alpha_k}$$

If $p \leq q$, the partial fraction expansion must include a polynomial in z^{-1} of order $(p-q)$. The coefficients of this polynomial may be found by long division (i.e., by dividing the numerator polynomial by the denominator). For multiple-order poles, the expansion must be modified. For example, if $X(z)$ has a second-order pole at $z = \alpha_k$, the expansion will include two terms,

$$\frac{B_1}{1 - \alpha_k z^{-1}} + \frac{B_2}{(1 - \alpha_k z^{-1})^2}$$

where B_1 , and B_2 are given by

$$B_1 = \alpha_k \left[\frac{d}{dz} (1 - \alpha_k z^{-1})^2 X(z) \right]_{z=\alpha_k}$$

$$B_2 = [(1 - \alpha_k z^{-1})^2 X(z)]_{z=\alpha_k}$$

- **Power Series**

The z-transform is a power series expansion,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \dots + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots$$

where the sequence values $x(n)$ are the coefficients of z^{-n} in the expansion. Therefore, if we can find the power series expansion for $X(z)$, the sequence values $x(n)$ may be found by simply picking off the coefficients of z^{-n} .

Example: Consider the z-transform

$$X(z) = \log(1 + az^{-1}) \quad |z| > |a|$$

Solution:

The power series expansion of this function is

$$\log(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{1}{n} (-1)^{n+1} a^n z^{-n}$$

Therefore, the sequence $x(n)$ having this z-transform is

$$x(n) = \begin{cases} \frac{1}{n} (-1)^{n+1} a^n & n > 0 \\ 0 & n \leq 0 \end{cases}$$

- **Contour Integration**

Another approach that may be used to find the inverse z-transform of $X(z)$ is to use contour integration. This procedure relies on Cauchy's integral theorem, which states that if C is a closed contour that encircles the origin in a counterclockwise direction,

$$\frac{1}{2\pi j} \oint_C z^{-k} dz = \begin{cases} 1 & k = 1 \\ 0 & k \neq 1 \end{cases}$$

With

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Cauchy's integral theorem may be used to show that the coefficients $x(n)$ may be found from $X(z)$ as follows:

$$x(n) = \frac{1}{2\pi j} \oint_C X(z)z^{n-1} dz$$

where C is a closed contour within the region of convergence of $X(z)$ that encircles the origin in a counterclockwise direction. Contour integrals of this form may often be evaluated with the help of Cauchy's residue theorem,

$$x(n) = \frac{1}{2\pi j} \oint_C X(z)z^{n-1} dz = \sum [\text{residues of } X(z)z^{n-1} \text{ at the poles inside } C]$$

If $X(z)$ is a rational function of z with a first-order pole at $z = \alpha_k$,

$$\text{Res}[X(z)z^{n-1} \text{ at } z = \alpha_k] = [(1 - \alpha_k z^{-1})X(z)z^{n-1}]_{z=\alpha_k}$$

Contour integration is particularly useful if only a few values of $x(n)$ are needed.

Example:

Find the inverse of each of the following z -transforms:

(a) $X(z) = 4 + 3(z^2 + z^{-2}) \quad 0 < |z| < \infty$

(b) $X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{3}{1 - \frac{1}{3}z^{-1}} \quad |z| > \frac{1}{2}$

(c) $X(z) = \frac{1}{1 + 3z^{-1} + 2z^{-2}} \quad |z| > 2$

(d) $X(z) = \frac{1}{(1 - z^{-1})(1 - z^{-2})} \quad |z| > 1$

- a) Because $X(z)$ is a finite-order polynomial, $x(n)$ is a finite-length sequence. Therefore, $x(n)$ is the coefficient that multiplies z^{-1} in $X(z)$. Thus, $x(0) = 4$ and $x(2) = x(-2) = 3$.
- b) This z -transform is a sum of two first-order rational functions of z . Because the region of convergence of $X(z)$ is the exterior of a circle, $x(n)$ is a right-sided sequence. Using the z -transform pair for a right-sided exponential, we may invert $X(z)$ easily as follows:

$$x(n) = \left(\frac{1}{2}\right)^n u(n) + 3\left(\frac{1}{3}\right)^n u(n)$$

- c) Here we have a rational function of z with a denominator that is a quadratic in z . Before we can find the inverse z -transform, we need to factor the denominator and perform a partial fraction expansion:

$$\begin{aligned} X(z) &= \frac{1}{1 + 3z^{-1} + 2z^{-2}} = \frac{1}{(1 + 2z^{-1})(1 + z^{-1})} \\ &= \frac{2}{1 + 2z^{-1}} - \frac{1}{1 + z^{-1}} \end{aligned}$$

Because $x(n)$ is right-sided, the inverse z -transform is

$$x(n) = 2(-2)^n u(n) - (-1)^n u(n)$$

- d) One way to invert this z -transform is to perform a partial fraction expansion. With

$$\begin{aligned} X(z) &= \frac{1}{(1 - z^{-1})(1 - z^{-2})} = \frac{1}{(1 - z^{-1})^2(1 + z^{-1})} \\ &= \frac{A}{1 + z^{-1}} + \frac{B_1}{1 - z^{-1}} + \frac{B_2}{(1 - z^{-1})^2} \end{aligned}$$

the constants A , B_1 , and B_2 are as follows:

$$A = [(1 + z^{-1})X(z)]_{z=-1} = \frac{1}{4}$$

$$B_1 = \left[\frac{d}{dz} (1 - z^{-1})^2 X(z) \right]_{z=1} = \left[\frac{z^{-2}}{(1 + z^{-1})^2} \right]_{z=1} = \frac{1}{4}$$

$$B_2 = [(1 - z^{-1})^2 X(z)]_{z=1} = \frac{1}{2}$$

Inverse transforming each term, we have

$$x(n) = \frac{1}{4} [(-1)^n + 1 + 2(n + 1)] u(n)$$

Example:

Find the inverse z-transform of the second-order system

$$X(z) = \frac{1 + \frac{1}{4}z^{-1}}{(1 - \frac{1}{2}z^{-1})^2} \quad |z| > \frac{1}{2}$$

Here we have a second-order pole at $z = \frac{1}{2}$. The partial fraction expansion for $X(z)$ is

$$X(z) = \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{(1 - \frac{1}{2}z^{-1})^2}$$

The constant A_1 is

$$A_1 = \frac{1}{2} \left[\frac{d}{dz} (1 - \frac{1}{2}z^{-1})^2 X(z) \right]_{z=1/2} = \frac{1}{2} \left[-\frac{1}{4}z^{-2} \right]_{z=1/2} = -\frac{1}{2}$$

and the constant A_2 is

$$A_2 = [(1 - \frac{1}{2}z^{-1})^2 X(z)]_{z=1/2} = \frac{3}{2}$$

Therefore,

$$X(z) = -\frac{\frac{1}{2}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{3}{2}}{(1 - \frac{1}{2}z^{-1})^2}$$

and

$$x(n) = -\left(\frac{1}{2}\right)^{n+1} u(n) + 3(n + 1)\left(\frac{1}{2}\right)^{n+1} u(n)$$

Example:

Find the inverse z-transform of $X(z) = \sin z$.

To find the inverse z-transform of $X(z) = \sin z$, we expand $X(z)$ in a Taylor series about $z = 0$ as follows:

$$\begin{aligned} X(z) &= X(z)\Big|_{z=0} + z \frac{dX(z)}{dz}\Big|_{z=0} + \frac{z^2}{2!} \frac{d^2X(z)}{dz^2}\Big|_{z=0} + \dots + \frac{z^n}{n!} \frac{d^n X(z)}{dz^n}\Big|_{z=0} + \dots \\ &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \end{aligned}$$

Because

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

we may associate the coefficients in the Taylor series expansion with the sequence values $x(n)$. Thus, we have

$$x(n) = (-1)^n \frac{1}{(2|n|+1)!} \quad n = -1, -3, -5, \dots$$

Example:

Evaluate the following integral:

$$\frac{1}{2\pi j} \oint_C \frac{1 + 2z^{-1} - z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{2}{3}z^{-1})} z^3 dz$$

where the contour of integration C is the unit circle.

Recall that for a sequence $x(n)$ that has a z-transform $X(z)$, the sequence may be recovered using contour integration as follows:

$$x(n) = \frac{1}{2\pi j} \oint_C X(z)z^{n-1} dz$$

Therefore, the integral that is to be evaluated corresponds to the value of the sequence $x(n)$ at $n = 4$ that has a z-transform

$$X(z) = \frac{1 + 2z^{-1} - z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{2}{3}z^{-1})}$$

Thus, we may find $x(n)$ using a partial fraction expansion of $X(z)$ and then evaluate the sequence at $n = 4$. With this approach, however, we are finding the values of $x(n)$ for all n . Alternatively, we could perform long division and divide the numerator of $X(z)$ by the denominator. The coefficient multiplying z^{-4} would then be the value of $x(n)$ at $n = 4$, and the value of the integral. However, because we are only interested in the value of the sequence at $n = 4$, the easiest approach is to evaluate the integral directly using the Cauchy integral theorem. The value of the integral is equal to the sum of the residues of the poles of $X(z)z^3$ inside the unit circle. Because

$$X(z)z^3 = z^3 \frac{z^2 + 2z - 1}{\left(z - \frac{1}{2}\right)\left(z - \frac{2}{3}\right)}$$

has poles at $z = 1/2$ and $z = 2/3$,

$$\text{Res}[X(z)z^3]_{z=\frac{1}{2}} = \left[z^3 \frac{z^2 + 2z - 1}{z - \frac{2}{3}} \right]_{z=\frac{1}{2}} = -\frac{3}{16}$$

and

$$\text{Res}[X(z)z^3]_{z=\frac{2}{3}} = \left[z^3 \frac{z^2 + 2z - 1}{z - \frac{1}{2}} \right]_{z=\frac{2}{3}} = \frac{112}{81}$$

Therefore, we have

$$\frac{1}{2\pi j} \oint_c X(z)z^3 dz = \frac{112}{81} - \frac{3}{16} = 1.1952$$

5.3. Ayırık Zamanlı Sistemlerin Z-Modelleri

First let us consider a discrete-time system as an interconnection of only three basic components: the delay elements, multipliers, and adders. The input–output relationships for these components and their symbols are shown in Figure 4.

The fourth component is the modulator, which multiplies two or more signals and hence performs a nonlinear operation.

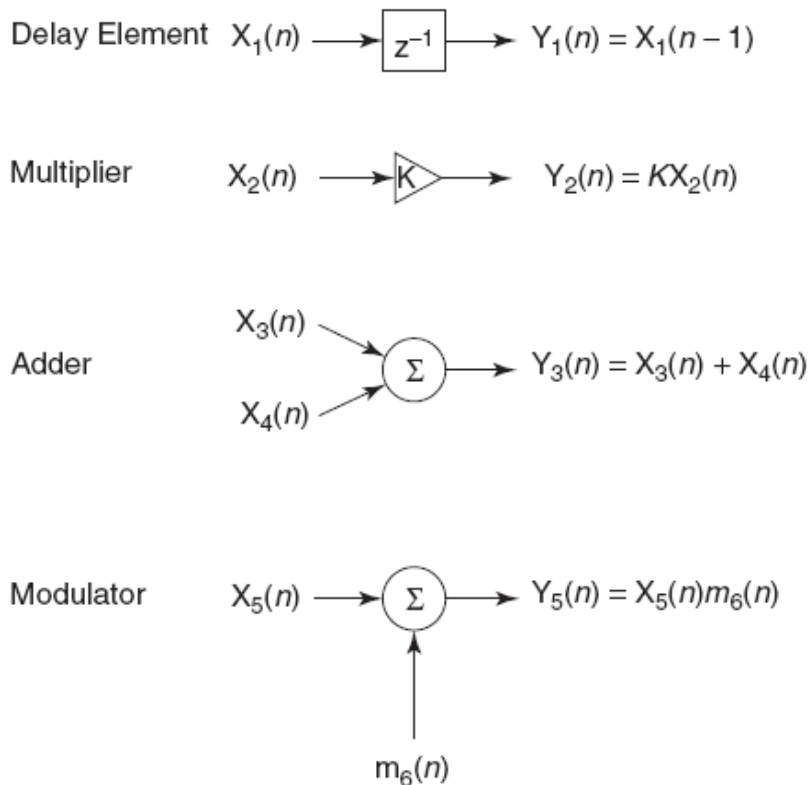


Figure The *basic components used in a discrete-time system.*

A simple discrete-time system is shown in Figure 5, where input signal $x(n) = \{x(0), x(1), x(2), x(3)\}$ is shown to the left of $v_0(n) = x(n)$. The signal $v_1(n)$ shown on the left is the signal $x(n)$ delayed by T seconds or one sample, so, $v_1(n) = x(n - 1)$. Similarly, v_2 and v_3 are the signals obtained from $x(n)$ when it is delayed by $2T$ and $3T$ seconds: $v_2(n) = x(n - 2)$ and $v_3(n) = x(n - 3)$. When we say that the signal $x(n)$ is delayed by T , $2T$, or $3T$ seconds, we mean that the samples of the sequence are present T , $2T$, or $3T$ seconds *later*, as shown by the plots of the signals to the left of $v_1(n)$, $v_2(n)$, and $v_3(n)$. But at any given time $t = nT$, the samples in $v_1(n)$, $v_2(n)$, and $v_3(n)$ are the samples of the input signal that occur T , $2T$, and $3T$ seconds *previous* to $t = nT$. For example, at $t = 3T$, the value of the sample in $x(n)$ is $x(3)$, and the values present in $v_1(n)$, $v_2(n)$ and $v_3(n)$ are $x(2)$, $x(1)$, and $x(0)$, respectively.

A good understanding of the operation of the discrete-time system as illustrated in Figure 4 is essential in analyzing, testing, and debugging the operation of the system when available software is used for the design, simulation, and hardware implementation of the system.

It is easily seen that the output signal in Figure 4 is

$$y(n) = b(0)v(0) + b(1)v(1) + b(2)v(2) + b(3)v(3)$$

$$= b(0)x(n) + b(1)x(n - 1) + b(2)x(n - 2) + b(3)x(n - 3)$$

where $b(0)$, $b(1)$, $b(2)$, $b(3)$ are the gain constants of the multipliers. It is also easy to see from the last expression that the output signal is the weighted sum of the current value and the previous three values of the input signal. So this gives us an input–output relationship for the system shown in Figure 4.

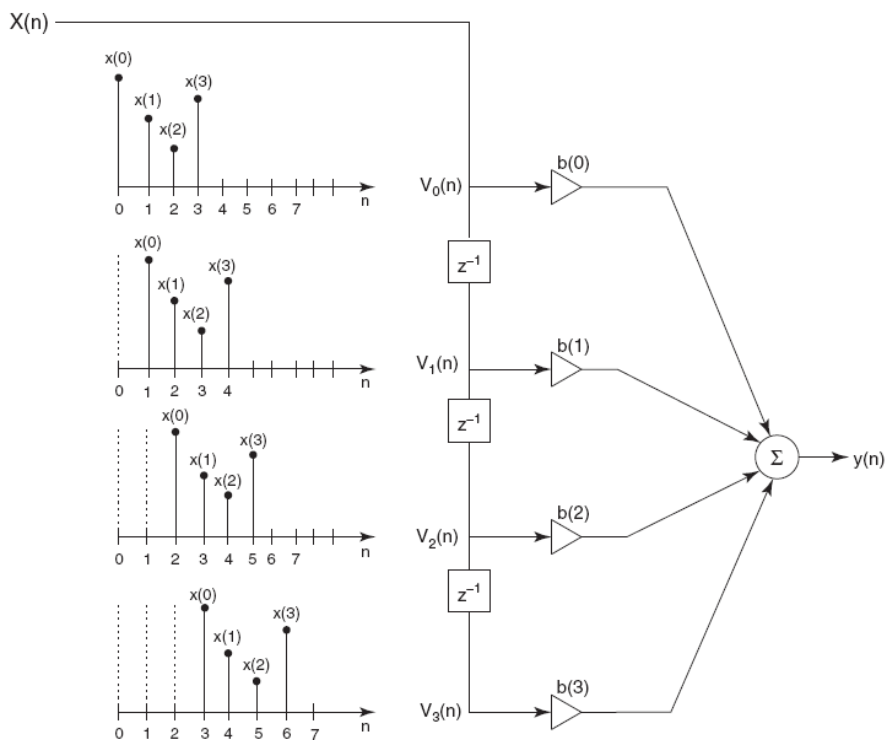


Figure. Operations in a typical discrete-time system.

Now we consider another example of a discrete-time system, shown in Figure 5. Note that a fundamental rule is to express the output of the adders and generate as many equations as the number of adders found in this circuit diagram for the discrete-time system. (This step is similar to writing the node equations for an analog electric circuit.) Denoting the outputs of the three adders as $y_1(n)$, $y_2(n)$, and $y_3(n)$, we get

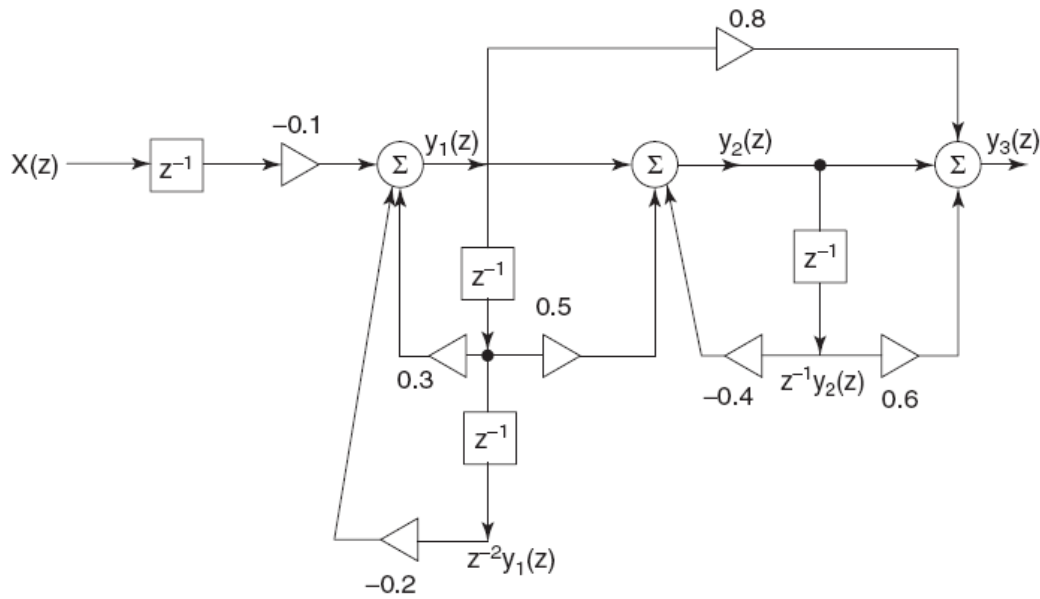


Figure. Schematic circuit for a discrete-time system.

$$y_1(n) = 0.3y_1(n - 1) - 0.2y_1(n - 2) - 0.1x(n - 1)$$

$$y_2(n) = y_1(n) + 0.5y_1(n - 1) - 0.4y_2(n - 1)$$

$$y_3(n) = y_2(n) + 0.6y_2(n - 1) + 0.8y_1(n)$$

These three equations give us a mathematical model derived from the model shown in Figure 5 that is schematic in nature. We can also derive (draw the circuit realization) the model shown in Figure 5 from the same equations given above. After eliminating the internal variables $y_1(n)$ and $y_2(n)$; that relationship constitutes the third model for the system. The general form of such an input–output relationship is

$$y(n) = - \sum_{k=1}^N a(k)y(n - k) + \sum_{k=0}^M b(k)x(n - k) \quad \text{Eq(1)}$$

or in another equivalent form

$$\sum_{k=0}^N a(k)y(n - k) = \sum_{k=0}^M b(k)x(n - k); \quad a(0) = 1 \quad \text{Eq(2)}$$

Eq(1) shows that the output $y(n)$ is determined by the weighted sum of the previous N values of the output and the weighted sum of the current and previous $M + 1$ values of the input. Very often the coefficient $a(0)$ as shown in Eq(2) is normalized to unity.

6. Hilbert –Huang Transform

The underlying principle of the Huang's EMD method is the concept of instantaneous frequency defined as the derivative of the phase of an analytic signal **Error! Reference source not found.** A mono-component signal, by definition, has a unique well-defined and positive instantaneous frequency represented by the derivative of the phase of the signal. A signal, with multiple modes of oscillation existing simultaneously, will not have a meaningful instantaneous frequency. Accordingly, a distorted signal must be decomposed into its constituent mono-component signals before the application of Hilbert transform to calculate the instantaneous frequency.

The essence of EMD is to recognize oscillatory modes existing in time scales defined by the interval between local extrema. A local extremum point is any point on the signal where its derivative is zero, and its second derivative non-zero. The term local is used to differentiate it from a global extremum point. For instance, within an observation window, there may be several local extrema, however only one global maximum and global minimum point may be present. Once the time scales are identified, IMFs with zero mean are sifted out of the signal. The steps comprising the EMD method are as follows:

- A1. Identify local maxima and minima of distorted signal, $s(t)$,
- A2. Perform cubic spline interpolation between the maxima and the minima to obtain the envelopes $e_M(t)$ and $e_m(t)$, respectively,

$$m(t) = \frac{(e_M(t) + e_m(t))}{2},$$

- A3. Compute mean of the envelopes,
- A4. Extract $c_1(t) = s(t) - m(t)$,
- A5. $c_1(t)$ is an IMF if the number of local extrema of $c_1(t)$, is equal to or differs from the number of zero crossings by one, AND the average of $c_1(t)$ reasonably zero. If $c_1(t)$ is not an IMF, then repeat steps A1-A4 on $c_1(t)$ instead of $s(t)$, until the new $c_1(t)$ obtained satisfies the conditions of an IMF,
- A6. Compute the residue, $r_1(t) = s(t) - c_1(t)$,
- A7. If the residue, $r_1(t)$, is above a threshold value of error tolerance, then repeat steps A1-A6 on $r_1(t)$, to obtain the next IMF and a new residue.

In practice, an appropriate stopping criterion, in step A5, avoids 'over-improving' $c_1(t)$ as that can lead to significant loss of information **Error! Reference source not found.** The first MF obtained, consists of the highest frequency components present in the original signal. The subsequent IMFs obtained, contain progressively lower frequency components of the signal. If n orthogonal IMFs are obtained in this iterative manner, the original signal may be reconstructed as,

$$s(t) = \sum_n c_i(t) + r(t) \quad (20.1)$$

The final residue exhibits any general trends followed by the original signal.

7. Convolution

Given two continuous time signals $x(t) = e^{-t}$ and $y(t) = e^{-2t}$ which exist for $t > 0$, the convolution $z(t) = x(t) * y(t)$ is

$$x(t) = e^{-t}$$

Laplace transformation

$$X(s) = \frac{1}{s+1}$$

$$y(t) = e^{-2t}$$

$$Y(s) = \frac{1}{s+2}$$

Convolution in time domain is equivalent to multiplication in frequency domain.

$$z(t) = x(t) * y(t)$$

$$Z(s) = X(s) Y(s) = \left(\frac{1}{s+1}\right)\left(\frac{1}{s+2}\right)$$

By partial fraction and taking inverse Laplace transformation, we get

$$Z(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$z(t) = e^{-t} - e^{-2t}$$

8. Sayısal Analiz

Laws and Rules of Boolean Algebra

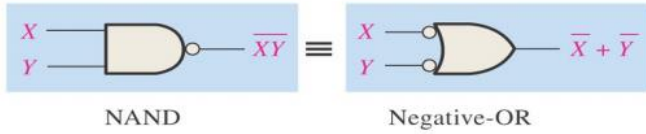
- Laws of Boolean Algebra
 - The 12 Rules of Boolean Algebra
 - $A + 0 = A$
 - $A + 1 = 1$
 - $A \cdot 0 = 0$
 - $A \cdot 1 = A$
 - $A + A = A$
 - $A + \bar{A} = 1$
 - $A \cdot A = A$
 - $A \cdot \bar{A} = 0$
 - $\overline{\bar{A}} = A$
 - $A + AB = A$
 - $A + \bar{A}B = A + B$
 - $(A + B)(A + C) = A + BC$

Laws and Rules of Boolean Algebra

- Laws of Boolean Algebra
 - Commutative Law
 - Commutative Law of Addition: $A + B = B + A$
 - Commutative Law of Multiplication: $AB = BA$
 - Associative Law
 - Associative Law of Addition: $A + (B + C) = (A + B) + C$
 - Associative Law of Multiplication: $A(BC) = (AB)C$
 - Distributive Law
 - $A(B + C) = AB + AC$

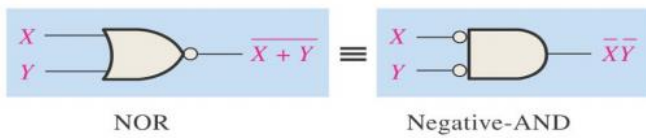
Demorgan's Theorems

$$\overline{XY} = \overline{X} + \overline{Y}$$



Inputs		Output	
X	Y	\overline{XY}	$\overline{X} + \overline{Y}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

$$\overline{X + Y} = \overline{X} \overline{Y}$$

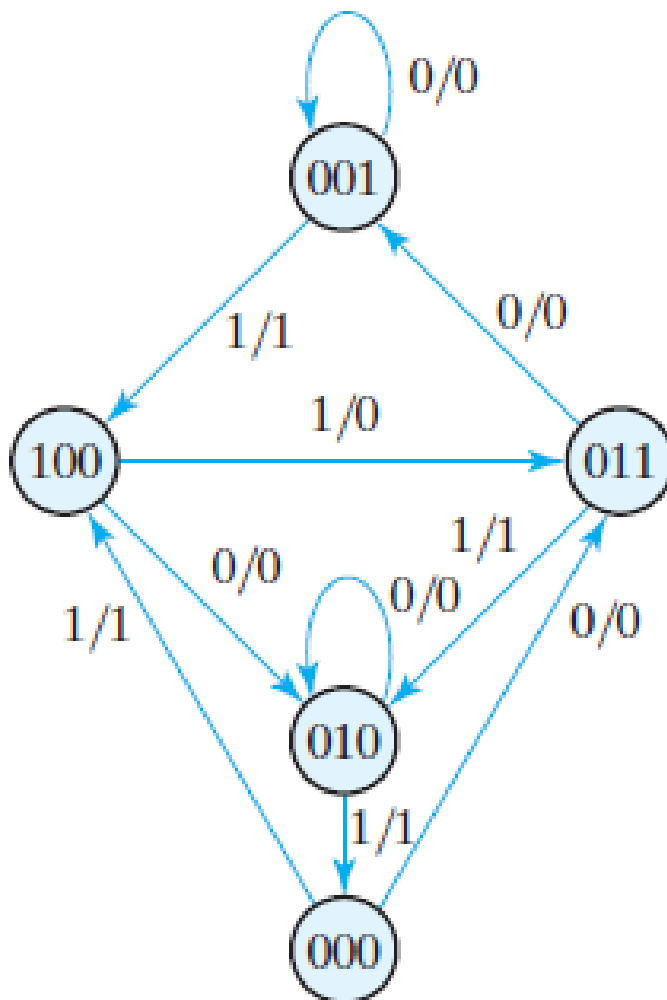


Inputs		Output	
X	Y	$\overline{X + Y}$	$\overline{X} \overline{Y}$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

Sequential logic has memory; the circuit stores the result of the previous set of inputs. The current output depends on inputs in the past as well as present inputs. The basic element in sequential logic is the bistable latch or flip-flop, which acts as a memory element for one bit of data.

FLIP-FLOP NAME	FLIP-FLOP SYMBOL	CHARACTERISTIC EQUATION	EXCITATION TABLE																				
SR		$Q_{(next)} = S + R'Q$ $SR = 0$	<table border="1"> <thead> <tr> <th>Q</th> <th>$Q_{(next)}$</th> <th>S</th> <th>R</th> </tr> </thead> <tbody> <tr> <td>0</td> <td>0</td> <td>0</td> <td>X</td> </tr> <tr> <td>0</td> <td>1</td> <td>1</td> <td>0</td> </tr> <tr> <td>1</td> <td>0</td> <td>0</td> <td>1</td> </tr> <tr> <td>1</td> <td>1</td> <td>X</td> <td>0</td> </tr> </tbody> </table>	Q	$Q_{(next)}$	S	R	0	0	0	X	0	1	1	0	1	0	0	1	1	1	X	0
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0	0	0	X																				
0	1	1	0																				
1	0	0	1																				
1	1	X	0																				
JK		$Q_{(next)} = JQ' + K'Q$	<table border="1"> <thead> <tr> <th>Q</th> <th>$Q_{(next)}$</th> <th>J</th> <th>K</th> </tr> </thead> <tbody> <tr> <td>0</td> <td>0</td> <td>0</td> <td>X</td> </tr> <tr> <td>0</td> <td>1</td> <td>1</td> <td>X</td> </tr> <tr> <td>1</td> <td>0</td> <td>X</td> <td>1</td> </tr> <tr> <td>1</td> <td>1</td> <td>X</td> <td>0</td> </tr> </tbody> </table>	Q	$Q_{(next)}$	J	K	0	0	0	X	0	1	1	X	1	0	X	1	1	1	X	0
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1	1	X	0																				
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Q	$Q_{(next)}$	T																					
0	0	0																					
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- *State table* – a multiple variable table with the following four sections:
 - *Present State* – the values of the state variables for each allowed state.
 - *Input* – the input combinations allowed.
 - *Next-state* – the value of the state at time (t+1) based on the present state and the input.
 - *Output* – the value of the output as a function of the present state and (sometimes) the input.
- From the viewpoint of a truth table:
 - the inputs are Input, Present State
 - and the outputs are Output, Next State



Durum diyagramı verilen devrenin

- Şuanki ve birsonraki durumlara göre D -ikili devre sayısını bulunur.
- Durum Tablosunu oluşturulur.
- Karnaugh Diyagramı ile indirgeyerek çıkış denklemleri bulunur.
- Devreyi çizilir.
- Yorumlanır.

9. Eklər

9.1. Matematiksel Kavramlar

Properties of Exponential and Logarithmic Equations

Let a be a positive real number such that $a \neq 1$, and let x and y be real numbers. Then the following properties are true:

1. $a^x = a^y$ if and only if $x = y$
2. $\log_a x = \log_a y$ if and only if $x = y$ ($x > 0, y > 0$)

Inverse Properties of Exponents and Logarithms

Base a	Natural Base e
1. $\log_a(a^x) = x$	$\ln(e^x) = x$
2. $a^{(\log_a x)} = x$	$e^{(\ln x)} = x$

$$4^{x+2} = 64 \quad \text{Original Equation}$$

$$4^{x+2} = 4^3 \quad \text{Rewrite with like bases}$$

$$x + 2 = 3 \quad \text{Property of exponential equations}$$

$$x = 1 \quad \text{Subtract 2 from both sides}$$

The solution is 1. Check this in the original equation.

$$\ln(2x - 3) = \ln 11 \quad \text{Original Equation}$$

$$2x - 3 = 11 \quad \text{Property of logarithmic equations}$$

$$2x = 14 \quad \text{Add 3 to both sides}$$

$$x = 7 \quad \text{Divide both sides by 2}$$

The solution is 7. Check this in the original equation.

$$5 + e^{x+1} = 20 \quad \text{Original Equation}$$

$$e^{x+1} = 15 \quad \text{Subtract 5 from both sides}$$

$$\ln e^{x+1} = \ln 15 \quad \text{Take the logarithm of both sides}$$

$$x + 1 = \ln 15 \quad \text{Inverse Property}$$

$$x = -1 + \ln 15 \approx 1.708 \quad \text{Subtract 1 from both sides}$$

$$5 + e^{x+1} = 20 \quad \text{Original Equation}$$

$$5 + e^{1.708+1} \stackrel{?}{=} 20 \quad \text{Substitute 1.708 for } x$$

$$5 + e^{2.708} \stackrel{?}{=} 20 \quad \text{Simplify}$$

$$5 + 14.999 \approx 20 \quad \text{Solution checks } \checkmark$$

$$2^x = 7 \quad \text{Original Equation}$$

$$\log 2^x = \log 7 \quad \text{Take the logarithm of both sides}$$

$$x(\log 2) = \log 7 \quad \text{Property of Logarithms}$$

$$x = \frac{\log 7}{\log 2} \approx 2.807 \quad \text{Solve for } x$$

$$4^{x-3} = 9 \quad \text{Original Equation}$$

$$\log 4^{x-3} = \log 9 \quad \text{Take the logarithm of both sides}$$

$$(x-3)\log 4 = \log 9 \quad \text{Property of Logarithms}$$

$$x-3 = \frac{\log 9}{\log 4} \quad \text{Divide both sides by } \log 4$$

$$x = 3 + \frac{\log 9}{\log 4} \approx 4.585 \quad \text{Solve for } x$$

$$2 \log_4 x = 5 \quad \text{Original Equation}$$

$$\log_4 x = \frac{5}{2} \quad \text{Divide both sides by 2}$$

$$4^{5/2} = x \quad \text{Change to exponential form}$$

$$x = 32 \quad \text{Simplify}$$

$$20 \ln 0.2x = 30 \quad \text{Original Equation}$$

$$\ln 0.2x = 1.5 \quad \text{Divide both sides by 20}$$

$$0.2x = e^{1.5} \quad \text{Change to exponential form}$$

$$x = 5e^{1.5} \approx 22.408 \quad \text{Divide both sides by 0.2}$$

$$2^x = 7 \quad \text{Original Equation}$$

$$\log_2 2^x = \log_2 7 \quad \text{Take the logarithm of both sides}$$

$$x = \log_2 7 \quad \text{Inverse Property}$$

$$x = \frac{\log 7}{\log 2} \approx 2.807 \quad \text{Change of Base Formula}$$

$$4^{x-3} = 9 \quad \text{Original Equation}$$

$$\log_4 4^{x-3} = \log_4 9 \quad \text{Take the logarithm of both sides}$$

$$x-3 = \log_4 9 \quad \text{Inverse Property}$$

$$x-3 = \frac{\log 9}{\log 4} \quad \text{Change of Base Formula}$$

$$x = 3 + \frac{\log 9}{\log 4} \approx 4.585 \quad \text{Solve for } x$$

$$3 \log x = 6 \quad \text{Original Equation}$$

$$\log x = 2 \quad \text{Divide both sides by 3}$$

$$10^2 = x \quad \text{Change to exponential form}$$

$$x = 100 \quad \text{Simplify}$$

$$\log_3 2x - \log_3(x-3) = 1 \quad \text{Original Equation}$$

$$\log_3 \frac{2x}{x-3} = 1 \quad \text{Condense the left side}$$

$$3^{\log_3 \frac{2x}{x-3}} = 3^1 \quad \text{Exponentiate both sides}$$

$$\frac{2x}{x-3} = 3 \quad \text{Inverse Property}$$

$$2x = 3x - 9 \quad \text{Multiply both sides by } x-3$$

$$x = 9 \quad \text{Solve for } x$$

- | | | | |
|---|---------------------|--------------------------------------|-------------------|
| 1. $3^{x-1} = 81$ | 1. 5 | 22. $3^{x-2} = 81$ | 22. 6 |
| 2. $8^x = 4$ | 2. $\frac{2}{3}$ | 23. $\log_3 x = 5$ | 23. 243 |
| 3. $e^x = 5$ | 3. 1.609 | 24. $\log_4 x = 3$ | 24. 64 |
| 4. $-14 + 3e^x = 11$ | 4. 2.120 | 25. $\log_2 2x = \log_2 100$ | 25. 50 |
| 5. $-6 + \ln 3x = 0$ | 5. 134.476 | 26. $\ln(x + 4) = \ln 7$ | 26. 3 |
| 6. $\log(3x + 1) = 2$ | 6. 33 | 27. $\log_3(2x + 1) = 2$ | 27. 4 |
| 7. $\ln x - \ln 3 = 4$ | 7. 163.794 | 28. $\log_5(x - 10) = 2$ | 28. 35 |
| 8. $2 \ln 3x = 4$ | 8. 2.463 | 29. $3^x = 500$ | 29. 5.66 |
| 9. $5^{x+2} = 4$ | 9. -1.139 | 30. $8^x = 1000$ | 30. 3.32 |
| 10. $\ln(x + 2)^2 = 6$ | 10. 18.086, -22.086 | 31. $\ln x = 7.25$ | 31. 1408.10 |
| 11. $4^{-3x} = 0.25$ | 11. $\frac{1}{3}$ | 32. $\ln x = -0.5$ | 32. 0.61 |
| 12. $2e^{2x} - 5e^x - 3 = 0$ | 12. 1.099 | 33. $2e^{0.5x} = 45$ | 33. 6.23 |
| 13. $\log_7 3 + \log_7 x = \log_7 32$ | 13. $\frac{32}{3}$ | 34. $100e^{-0.6x} = 20$ | 34. 2.68 |
| 14. $2 \log_6 4x = 0$ | 14. $\frac{1}{4}$ | 35. $12(1 - 4^x) = 18$ | 35. No Solution |
| 15. $\log_2 x + \log_2(x - 3) = 2$ | 15. 4 | 36. $25(1 - e^t) = 12$ | 36. -0.65 |
| 16. $\log_2(x + 5) - \log_2(x - 2) = 3$ | 16. 3 | 37. $\log 2x = 1.5$ | 37. 15.81 |
| 17. $4 \ln(2x + 3) = 11$ | 17. 6.321 | 38. $\log_2 2x = -0.65$ | 38. 0.32 |
| 18. $\log x - \log 6 = 2 \log 4$ | 18. 96 | 39. $\frac{1}{3} \log_2 x + 5 = 7$ | 39. 64 |
| 19. $2^x = 64$ | 19. 6 | 40. $4 \log_5(x + 1) = 4.8$ | 40. 5.90 |
| 20. $5^x = 25$ | 20. 2 | 41. $\log_2 x + \log_2 3 = 3$ | 41. $\frac{8}{3}$ |
| 21. $4^{x-3} = \frac{1}{16}$ | 21. 1 | 42. $2 \log_4 x - \log_4(x - 1) = 1$ | 42. 2 |

Binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

If n is a positive integer the series terminates and is valid for all x : the term in x^r is ${}^nC_r x^r$ or $\binom{n}{r}$ where ${}^nC_r \equiv \frac{n!}{r!(n-r)!}$ is the number of different ways in which an unordered sample of r objects can be selected from a set of n objects without replacement. When n is not a positive integer, the series does not terminate: the infinite series is convergent for $|x| < 1$.

Taylor and Maclaurin Series

If $y(x)$ is well-behaved in the vicinity of $x = a$ then it has a Taylor series,

$$y(x) = y(a+u) = y(a) + u \frac{dy}{dx} + \frac{u^2}{2!} \frac{d^2y}{dx^2} + \frac{u^3}{3!} \frac{d^3y}{dx^3} + \dots$$

where $u = x - a$ and the differential coefficients are evaluated at $x = a$. A Maclaurin series is a Taylor series with $a = 0$,

$$y(x) = y(0) + x \frac{dy}{dx} + \frac{x^2}{2!} \frac{d^2y}{dx^2} + \frac{x^3}{3!} \frac{d^3y}{dx^3} + \dots$$

Power series with real variables

e^x	$= 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$	valid for all x
$\ln(1+x)$	$= x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n+1} \frac{x^n}{n} + \dots$	valid for $-1 < x \leq 1$
$\cos x$	$= \frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	valid for all values of x
$\sin x$	$= \frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$	valid for all values of x
$\tan x$	$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$	valid for $-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\tan^{-1} x$	$= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$	valid for $-1 \leq x \leq 1$
$\sin^{-1} x$	$= x + \frac{1}{2} \frac{x^3}{3} + \frac{1.3}{2.4} \frac{x^5}{5} + \dots$	valid for $-1 < x < 1$

Integer series

$$\sum_1^N n = 1 + 2 + 3 + \dots + N = \frac{N(N+1)}{2}$$

$$\sum_1^N n^2 = 1^2 + 2^2 + 3^2 + \dots + N^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_1^N n^3 = 1^3 + 2^3 + 3^3 + \dots + N^3 = [1 + 2 + 3 + \dots + N]^2 = \frac{N^2(N+1)^2}{4}$$

$$\sum_1^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2$$

$$\sum_1^{\infty} \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

$$\sum_1^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}$$

$$\sum_1^N n(n+1)(n+2) = 1.2.3 + 2.3.4 + \dots + N(N+1)(N+2) = \frac{N(N+1)(N+2)(N+3)}{4}$$

This last result is a special case of the more general formula,

$$\sum_1^N n(n+1)(n+2)\dots(n+r) = \frac{N(N+1)(N+2)\dots(N+r)(N+r+1)}{r+2}.$$

Trigonometric Formula

$$\cos^2 A + \sin^2 A = 1$$

$$\sec^2 A - \tan^2 A = 1$$

$$\operatorname{cosec}^2 A - \cot^2 A = 1$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos A \cos B = \frac{\cos(A + B) + \cos(A - B)}{2}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin A \sin B = \frac{\cos(A - B) - \cos(A + B)}{2}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A \cos B = \frac{\sin(A + B) + \sin(A - B)}{2}$$

$$\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}$$

$$\cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}$$

$$\cos^3 A = \frac{3 \cos A + \cos 3A}{4}$$

$$\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}$$

$$\sin^3 A = \frac{3 \sin A - \sin 3A}{4}$$

Differentiation

$$(uv)' = u'v + uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \dots + {}^nC_r u^{(n-r)}v^{(r)} + \dots + uv^{(n)}$$

$$\text{where } {}^nC_r \equiv \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\operatorname{coth} x) = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \operatorname{coth} x$$

Temel İntegral

$$1) \int a \, dx = ax + c$$

$$2) \int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

$$3) \int \frac{1}{x} \, dx = \ln|x| + c$$

$$4) \int e^x \, dx = e^x + c$$

$$5) \int a^x \, dx = \frac{a^x}{\ln|a|} + c$$

$$6) \int \sin x \, dx = -\cos x + c$$

$$7) \int \cos x \, dx = \sin x + c$$

Integration

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad \text{for } n \neq -1$$

$$\int \frac{1}{x} dx = \ln x + c \quad \int \ln x dx = x(\ln x - 1) + c$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + c \quad \int x e^{ax} dx = e^{ax} \left(\frac{x}{a} - \frac{1}{a^2} \right) + c$$

$$\int x \ln x dx = \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + c$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{a} \tanh^{-1} \left(\frac{x}{a} \right) + c = \frac{1}{2a} \ln \left(\frac{a+x}{a-x} \right) + c \quad \text{for } x^2 < a^2$$

$$\int \frac{1}{x^2 - a^2} dx = -\frac{1}{a} \coth^{-1} \left(\frac{x}{a} \right) + c = \frac{1}{2a} \ln \left(\frac{x-a}{x+a} \right) + c \quad \text{for } x^2 > a^2$$

$$\int \frac{x}{(x^2 \pm a^2)^n} dx = \frac{-1}{2(n-1)} \frac{1}{(x^2 \pm a^2)^{n-1}} + c \quad \text{for } n \neq 1$$

$$\int \frac{x}{x^2 \pm a^2} dx = \frac{1}{2} \ln(x^2 \pm a^2) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left(x + \sqrt{x^2 \pm a^2} \right) + c$$

$$\int e^x dx = e^x$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}$$

$$\int b^{ax} dx = \frac{1}{a \ln(b)} b^{ax} \quad ; b > 0$$

$$\int \ln(x) dx = x \ln(x) - x$$

$$\int a^x \ln(a) dx = a^x \quad ; a > 0$$

Dirac δ -'function'

$$\delta(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\omega(t - \tau)] d\omega.$$

If $f(t)$ is an arbitrary function of t then $\int_{-\infty}^{\infty} \delta(t - \tau) f(t) dt = f(\tau)$.

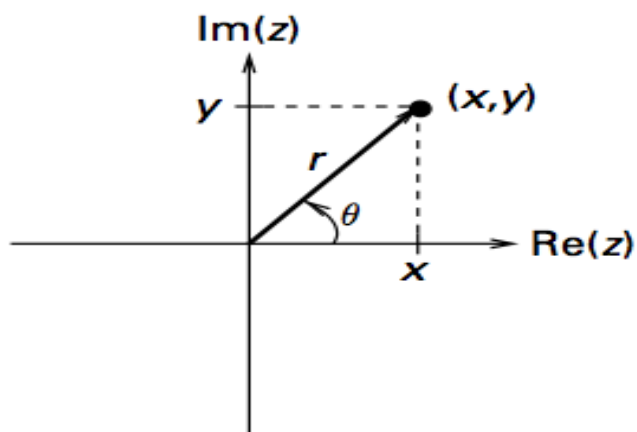
$\delta(t) = 0$ if $t \neq 0$, also $\int_{-\infty}^{\infty} \delta(t) dt = 1$

Complex

Complex number $z = x + jy$ (x and y real-valued; $j = \sqrt{-1}$)

$$r = |z| = \sqrt{x^2 + y^2},$$

$$\theta = \arg(z) = \tan^{-1} \frac{y}{x}$$



$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$e^{i\pi} + 1 = \text{Cos}(180) + i\text{Sin}(180) = -1 + 1 = 0$$

Aşağıdaki ifadeleri basitleştirin. Cevaplarınızı hem polar hem de dikdörtgen şeklinde verin.

$$a) \quad c = 3e^{j\pi/4} + 4e^{-j\pi/2}$$

$$b) \quad c = (-1 + 2j)^5$$

$$c) \quad c = 2e^{j\pi/2} - 3e^{j\pi/3}$$

$$\begin{aligned} | a) \quad c &= 3e^{j\pi/4} + 4e^{-j\pi/2} \\ &= 3 \cos \pi/4 + 3j \sin \pi/4 + 4 \cos(-\pi/2) + 4j \sin(-\pi/2) \\ &= \frac{3\sqrt{2}}{2} + j\frac{3\sqrt{2}}{2} - 4j \end{aligned}$$

$$c = 2.121 - 1.878j = 2.833 e^{j^{-0.725}}$$

$$b) \quad c = (-1 + 2j)^5 = [2.236 e^{2.034j}]^5$$

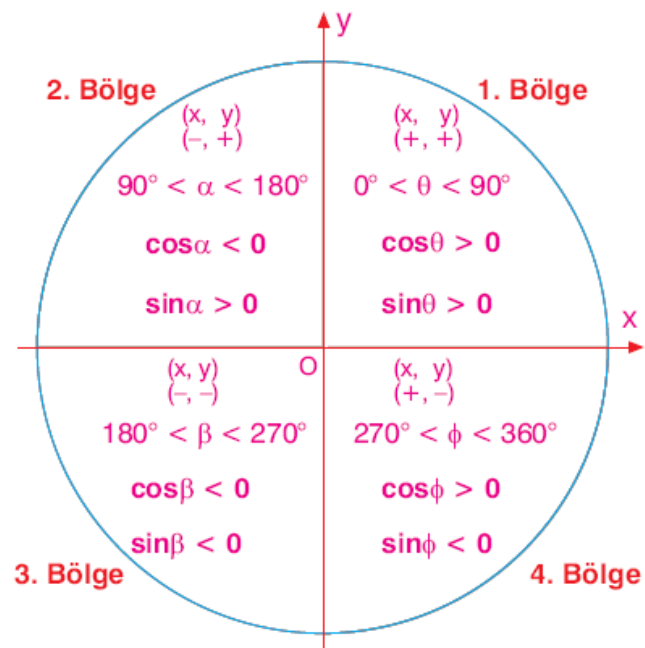
$$\begin{aligned} &= 2.236^5 e^{j10.172} \quad \text{Subtr } 2\pi \text{ off of angle} \\ &= 55.9 e^{j10.172} \\ &= 55.9 e^{j3.889} = -41 - 38j \end{aligned}$$

$$c) \quad c = 2e^{j\pi/2} - 3e^{j\pi/3}$$

$$\begin{aligned} &= 2 \cos \pi/2 + 2j \sin \pi/2 - 3 \cos \pi/3 - 3j \sin \pi/3 \\ &= 2j - \frac{3}{2} - 2.598j \end{aligned}$$

$$= -1.5 - j0.598 = 1.615 e^{3.521j}$$

Trigonometri



	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Tanımsız

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos \theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$$

$$\cos x = \frac{1}{2} (e^{ix} + e^{-ix})$$

$$\sin x = \frac{1}{2i} (e^{ix} - e^{-ix})$$

$$\sin(\pi - \theta) = \sin \theta$$

$$\sin(\pi + \theta) = -\sin \theta$$

$$\cos(\pi - \theta) = -\cos \theta$$

$$\cos(\pi + \theta) = -\cos \theta$$

Sine and Cosine Addition and Subtraction Formulas

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

$$\sin(a - b) = \sin a \cos b - \cos a \sin b$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

Limit

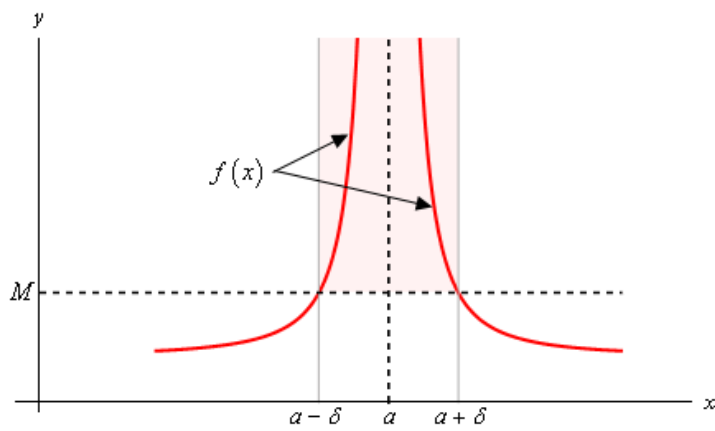
Soru:

$$F_1(t) = 1 - \exp(-t),$$

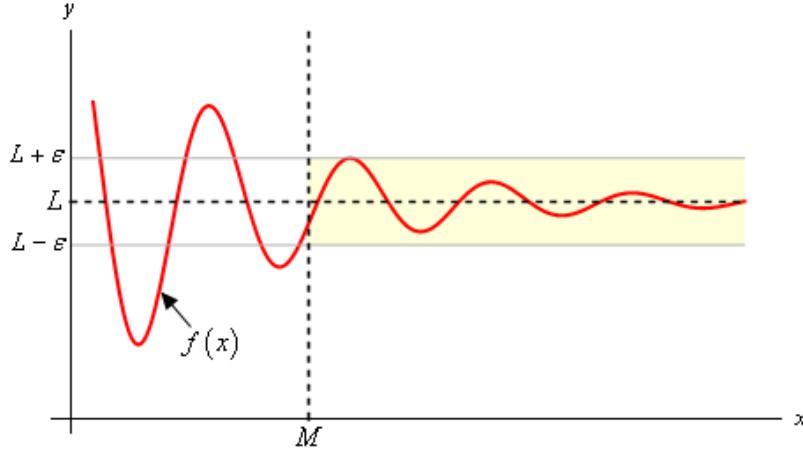
$$F_2(t) = 1 - k \cdot \exp(-k \cdot t)$$

Yukarı verilen sinyallerin sonsuza eğiliminde limitlerini bulun.

Yanıt: her iki durumda da 1 elde ederiz.



$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$



$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

10. Kaynaklar

- Basics of Signals and Systems, Gloria Menegaz, AA 2011-2012
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- A Practical Approach To Signals And Systems, D. Sundararajan, John Wiley & Sons (Asia) Pte Ltd. 2008.

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Sincerely,
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